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LINEAR COMBINATIONS OF STARLIKE AND CONVEX FUNCTIONS OF ORDER α WITH COMPLEX COEFFICIENTS

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1. INTRODUCTION.

Let A denote the class of functions f(z), which are analytic in the unit disk U and normalized by f(0)=0, f'(0)=1. For $0 \leqq \alpha < 1$, we define the subclasses of A,

$$S^*(\alpha) = \{f(z) \in A;\ \mathrm{Re}\{\frac{zf'(z)}{f(z)}\}>\alpha,\ z \in U\}, \tag{1.1}$$

$$K(\alpha) = \{f(z) \in A;\ \mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\}>\alpha,\ z \in U\}. \tag{1.2}$$

These are called the classes of starlike functions of order α and of convex functions of order α, respectively. It is easily seen that these are decreasing functions with respect to α and $S^*(\alpha) \supset K(\alpha)$. Let S be the class of normalized analytic and univalent functions in U. Then we know $K \subset S^* \subset S$, $K=K(0)$, $S^*=S^*(0)$.

Let us define another class $P(\alpha)$ for $\alpha<1$.

$$P(\alpha)=\{p(z);\ \text{analytic in } U,\ p(0)=1,\ \mathrm{Re}\{p(z)\}>\alpha,\ z \in U\}. \tag{1.3}$$

The class has a well-known property by the subordination principle.

LEMMA 1. If $p(z) \in P(\alpha)$ for $z \in U$, then

$$|p(z) - \frac{1+(1-2\alpha)|z|^2}{1-|z|^2}| \leqq \frac{2(1-\alpha)|z|^2}{1-|z|^2}. \tag{1.4}$$

It holds the equality of (1.4) by $p(z) = \frac{1+(1-2\alpha)z}{1-z}$.

If $f(z) \in S^*(\alpha)$, then $\frac{zf'(z)}{f(z)} = p(z) \in P(\alpha)$ and $p(z)$ is subordinate to $\frac{1+(1-2\alpha)z}{1-z}$, namely, we can write $\frac{zf'(z)}{f(z)} = \frac{1+(1-2\alpha)w(z)}{1-w(z)}$, where $w(z)$ is analytic in U with $w(0)=0$, $|w(z)|<1$, $z \in U$.

LEMMA 2.[2] If $f(z) \in S^*(\alpha)$, then

$$|\arg \frac{f(z)}{z}| \leqq 2(1-\alpha)\mathrm{Sin}^{-1}|z|, \quad z \in U. \tag{1.5}$$

It holds the equality of (1.5) by $f(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$.

We have the following lemma from (1.5) and the fact that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$.

LEMMA 3. If $f(z) \in K(\alpha)$, then

$$|\arg f'(z)| \leqq 2(1-\alpha)\mathrm{Sin}^{-1}|z|, \quad z \in U. \tag{1.6}$$

It holds the equality of (1.6) by

$$f(z) = \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & (\alpha \neq \frac{1}{2}) \\ -\log(1-z) & (\alpha = \frac{1}{2}). \end{cases} \tag{1.7}$$

The following result has been shown by T.H.MacGregor[1] and completely proved by D.R.Wilken and J.Feng[4].

LEMMA 4. If $f(z) \in K(\alpha)$, then $f(z) \in S^*(\beta(\alpha))$, where

$$(1.8) \qquad \beta(\alpha) = \begin{cases} \dfrac{2\alpha - 1}{2(1-2^{1-2\alpha})} & (\alpha \neq \frac{1}{2}) \\ \dfrac{1}{2 \log 2} & (\alpha = \frac{1}{2}). \end{cases}$$

2. THEOREMS AND PROOFS.

Lemma 5 is due to R.K.Stump[3] for which we will give a simple computation.

LEMMA 5. If $|u-a| \leqq d$, $|v-a| \leqq d$, where both u and v are complex variables and a and d are real constants such that $a>d>0$, moreover, for $\rho>0$ and $\theta \in [0,\pi)$ letting

$$(2.1) \qquad W = u\cdot\frac{1}{1+\rho e^{i\theta}} + v\cdot\frac{1}{1+\rho^{-1}e^{-i\theta}} .$$

Then

$$(2.2) \qquad \mathrm{Re}\, W \geqq a - d\cdot\sec(\frac{\theta}{2}).$$

Proof. The condition $|u-a| \leqq d$ implies $|uu_0-au_0| \leqq d|u_0|$ for some complex number u_0. This shows $\mathrm{Re}(uu_0) \geqq \mathrm{Re}(au_0) - d|u_0|$. At the same time, for a complex number v_0, $\mathrm{Re}(vv_0) \geqq \mathrm{Re}(av_0) - d|v_0|$.

Now, putting

(2.3) $$u_0 = \frac{1}{1+\rho e^{i\theta}}, \quad v_0 = \frac{1}{1+\rho^{-1}e^{-i\theta}},$$

it holds that

(2.4) $$\text{Re } W = \text{Re}(uu_0) + \text{Re}(vv_0)$$
$$\geqq a\{\text{Re}(u_0) + \text{Re}(v_0)\} - d(|u_0|+|v_0|).$$

It follows from (2.3) that

$$\text{Re}(u_0) = \frac{1 + \rho\cos\theta}{1+\rho^2+2\rho\cos\theta}, \quad \text{Re}(v_0) = \frac{\rho^2 + \rho\cos\theta}{1+\rho^2+2\rho\cos\theta},$$

(2.5) $$\text{Re}(u_0) + \text{Re}(v_0) = 1.$$

Also, $$|u_0| = \frac{1}{(1+\rho^2+2\rho\cos\theta)^{1/2}}, \quad |v_0| = \frac{\rho}{(1+\rho^2+2\rho\cos\theta)^{1/2}},$$

(2.6) $$|u_0| + |v_0| = \frac{1 + \rho}{(1+\rho^2+2\rho\cos\theta)^{1/2}} \leqq \sec(\frac{\theta}{2}).$$

Therefoe, we have Re $W \geqq a - d\cdot\sec(\frac{\theta}{2})$. This completes the proof.

THEOREM 1. For a complex number λ, let $f_1(z) \in S^*(\alpha)$, $f_2(z) \in S^*(\alpha)$ and

(2.7) $$F(z) = \lambda f_1(z) + (1-\lambda)f_2(z), \quad z \in U,$$

where $0 \leqq \delta = \arg \frac{\lambda}{1-\lambda} < \pi$. Then $F(z)$ is the starlike function of order μ for $|z| < \min\{\sin \frac{\pi - \delta}{4(1-\alpha)}, r_s\}$, where r_s is the smallest positive root of the equation,

(2.8) $$\frac{1+(1-2\alpha)r^2}{1 - r^2} - \frac{2(1-\alpha)r}{1 - r^2}\cdot\sec\{\frac{\delta}{2} + 2(1-\alpha)\text{Sin}^{-1}r\} = \mu.$$

Proof. From (2.7) we have

$$\frac{zF'(z)}{F(z)} = \frac{zf_2'(z)}{f_2(z)}\cdot[1+\{\frac{\lambda}{1-\lambda}\cdot\frac{f_1(z)}{f_2(z)}\}]^{-1} + \frac{zf_1'(z)}{f_1(z)}\cdot[1+\{\frac{\lambda}{1-\lambda}\cdot\frac{f_1(z)}{f_2(z)}\}^{-1}]^{-1}. \tag{2.9}$$

Putting in Lemma 5 $u = \frac{zf_2'(z)}{f_2(z)}$, $v = \frac{zf_1'(z)}{f_1(z)}$,

$a = \frac{1+(1-2\alpha)|z|^2}{1-|z|^2}$, $d = \frac{2(1-\alpha)|z|}{1-|z|^2}$ and $\rho e^{i\theta} = \frac{\lambda}{1-\lambda}\cdot\frac{f_1(z)}{f_2(z)}$,

it follows from the result of Lemma 5

$$\text{Re}\{\frac{zF'(z)}{F(z)}\} \geqq \frac{1+(1-2\alpha)|z|^2}{1-|z|^2} - \frac{2(1-\alpha)|z|}{1-|z|^2}\cdot\sec(\frac{\gamma}{2}), \tag{2.10}$$

where

$$\gamma = \arg\frac{\lambda}{1-\lambda}\cdot\frac{f_1(z)}{f_2(z)} = \delta + \arg\frac{f_1(z)}{z} - \arg\frac{f_2(z)}{z}.$$

Hence, providing $|\gamma| \leqq \delta + 4(1-\alpha)\text{Sin}^{-1}|z|$ from Lemma 2, $z \in U$ and $|z| \leqq r < \sin\frac{\pi-\delta}{4(1-\alpha)}$, then $0 \leqq |\gamma| < \pi$ and $\sec(\frac{\gamma}{2}) > 0$. Now if we let r_s be the smallest positive root of (2.8), then $\text{Re}\{\frac{zF'(z)}{F(z)}\} > \mu$ for $|z| < \min\{\sin\frac{\pi-\delta}{4(1-\alpha)}, r_s\}$. This completes the proof.

THEOREM 2. Under the same notations as in Theorem 1, let $f_1(z) \in K(\alpha)$, $f_2(z) \in K(\alpha)$, and $F(z) = \lambda f_1(z) + (1-\lambda)f_2(z)$, $z \in U$. Then $F(z)$ is the convex function of order μ for $|z| < \min\{\sin\frac{\pi-\delta}{4(1-\alpha)}, r_c\}$, where r_c is the smallest positive root of (2.8).

Proof. We have from the definition of $F(z)$,

$$(2.11) \qquad 1+\frac{zF''(z)}{F'(z)} = \{1+\frac{zf_1''(z)}{f_1'(z)}\}[1+\{\frac{\lambda}{1-\lambda}\cdot\frac{f_1'(z)}{f_2'(z)}\}^{-1}]^{-1} + \{1+\frac{zf_2''(z)}{f_2'(z)}\}[1+\{\frac{\lambda}{1-\lambda}\cdot\frac{f_1'(z)}{f_2'(z)}\}]^{-1}.$$

Also, it holds from Lemma 5 that

$$(2.12) \qquad \mathrm{Re}\{1+\frac{zF''(z)}{F'(z)}\} \geqq \frac{1+(1-2\alpha)|z|^2}{1-|z|^2} - \frac{2(1-\alpha)|z|}{1-|z|^2}\cdot\sec(\frac{\gamma}{2}),$$

where $\gamma = \arg\{\frac{\lambda}{1-\lambda}\cdot\frac{f_1'(z)}{f_2'(z)}\} = \delta + \arg f_1'(z) - \arg f_2'(z)$.

Hence, providing $|\gamma| \leqq \delta + 4(1-\alpha)\mathrm{Sin}^{-1}|z|$ after using Lemma 3, then $0 \leqq |\gamma| < \pi$ and $\sec(\frac{\gamma}{2}) > 0$ for $|z| < \sin\frac{\pi-\delta}{4(1-\alpha)}$. Therefore, $\mathrm{Re}\{1+\frac{zF''(z)}{F'(z)}\} > \mu$ for $|z| < \min\{\sin\frac{\pi-\delta}{4(1-\alpha)}, r_c\}$, where r_c is the smallest positive root of (2.8).

Theorem 3 follows by the above methods, therefore we omit the proof.

THEOREM 3. Let $f_1(z) \in K(\alpha)$, $f_2(z) \in K(\alpha)$ and $F(z) = \lambda f_1(z) + (1-\lambda)f_2(z)$, $z \in U$, where $0 \leqq \delta = \arg\frac{\lambda}{1-\lambda} < \pi$. Then $F(z)$ is the starlike function of order μ for $|z| < \min\{\sin\frac{\pi-\delta}{4(1-\beta(\alpha))}, r_s\}$, where $\beta(\alpha)$ is denoted in Lemma 4, and r_s is the smallest positive root of

$$(2.13) \qquad \frac{1+(1-2\alpha)r^2}{1-r^2} - \frac{2(1-\beta(\alpha))r}{1-r^2}\cdot\sec\{\frac{\gamma}{2} + 2(1-\beta(\alpha))\cdot\mathrm{Sin}^{-1}r\} = \mu.$$

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