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On certain multivalent functions

By

Mamoru NUNOKAWA (Gunma Univ.) (群馬大・教育　布川護)

Let A(p) be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in N = \{1,2,3,\dots\})$$

which are analytic in $D = \{ z \mid |z| < 1 \}$.

A function f(z) in A(p) is said to be p-valently starlike iff

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0 \qquad \text{in } D.$$

We denote by S(p) the subclass of A(p) consisting of functions which are p-valently starlike in D.

A function f(z) in A(p) is said to be p-valently close-to-convex iff there exists a function $g(z) \in S(p)$ such that

$$\mathrm{Re}\,\frac{zf'(z)}{g(z)} > 0 \qquad \text{in } D.$$

We denote by K(p) the subclass of A(p) consisting of functions which are p-valently close-to-convex in D.

Livingston [2] introduced this class K(p).

Ozaki [5, Theorem 3] proved that if $f(z) \in A(p)$ and

$$1 + \mathrm{Re}\,\frac{zf''(z)}{f'(z)} < \frac{k+p+1}{2} \qquad \text{in } D,$$

then f(z) is at most k-valent in D.

Moreover, by using Umezawa's result [7, Theorem 6] we can prove that if $f(z) \in A(p)$ and

$$1 + \mathrm{Re}\,\frac{zf''(z)}{f'(z)} < \frac{k+p+1}{2} \qquad \text{in } D,$$

then f(z) is convex of order at most p in one direction in D, and at most k-valent in D.

Nunokawa and Owa [4, Theorem 2] proved the following result.

LEMMA 1. Let $f(z) \in A(p)$ and

$$1 + \mathrm{Re}\,\frac{zf''(z)}{f'(z)} < \beta \qquad \text{in } D$$

where $p < \beta \leqq p + \frac{1}{2}$

Then we have

$$\left| \arg \frac{f'(z)}{z^{p-1}} \right| \leqq 2(\beta - p)\,\mathrm{Sin}^{-1}|z| \qquad \text{in } D$$

and therefore

$$\mathrm{Re}\,\frac{f'(z)}{z^{p-1}} > 0 \qquad \text{in } D.$$

This shows that $f(z)$ is p-valently close-to-convex or $f(z) \in K(p)$.

On the other hand, Nunokawa [3] proved the following result.

LEMMA 2. Let $f(z) \in A(p)$ and assume that

$$\mathrm{Re}\,\frac{f'(z)}{z^{p-1}} > 0 \qquad \text{in } D$$

and

$$\left(\mathrm{Im}\,\frac{f'(z)}{z^{p-1}} \right)\left(\mathrm{Im}\, e^{-i\alpha} z \right) \neq 0, \qquad z \in D(\alpha)$$

for some α ($0 \leqq \alpha < \pi$), where

$$D(\alpha) = \left\{ z \mid 0 < |z| < 1 \text{ and } (\arg z - \alpha)(\arg z - \pi - \alpha) \neq 0 \right\}.$$

Then $f(z)$ is p-valently starlike in D, $f(z) \in S(p)$ or

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0 \qquad \text{in } D.$$

Applying LEMMA 1 and 2, we can prove the following theorem.

THEOREM 1. Let $f(z) \in A(p)$ and assume that

$$1 + \mathrm{Re}\,\frac{zf''(z)}{f'(z)} < p + \frac{1}{2} \qquad \text{in } D$$

and

$$\left(\mathrm{Im}\,\frac{f'(z)}{z^{p-1}} \right)\left(\mathrm{Im}\, e^{-i\alpha} z \right) \neq 0 \qquad z \in D(\alpha)$$

for some α ($0 \leqq \alpha < \pi$) where

$$D(\alpha) = \{ z \mid 0 < |z| < 1 \text{ and } (\arg z - \alpha)(\arg z - \pi - \alpha) \neq 0 \} .$$

Then f(z) is p-valently starlike in D, $f(z) \in S(p)$ or

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \qquad \text{in } D.$$

PROOF. Applying LEMMA 1 to f(z), we have

$$\mathrm{Re}\, \frac{f(z)}{z^{p-1}} > 0 \qquad \text{in } D.$$

From the assumption of THEOREM 1 and LEMMM 2, it follows that

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \qquad \text{in } D.$$

This completes our proof.

From THEOREM 1, we easily have the following corollary.

COROLLARY 1. Let $f(z) \in A(p)$ and assume that

$$1 + \mathrm{Re}\, \frac{zf''(z)}{f'(z)} < p + 1 \qquad \text{in } D$$

and $f'(z)/z^{p-1}$ is typically real in D.

Then f(z) is p-valently starlike in D or $f(z) \in S(p)$.

THEOREM 2. Let $f(z) \in A(p)$ and assume that

(1) $$p - \frac{2p(\beta - p)}{k - p} < 1 + \mathrm{Re}\, \frac{zf''(z)}{f'(z)} < \beta \qquad \text{in } D$$

where $1 + p \leqq k$ and $(p + k - 1)/2 < \beta \leqq (p + k + 1)/2$.

Then f(z) is p-valently close-to-convex in D or $f(z) \in K(p)$.

PROOF. Let us put

(2) $$H(z) = \frac{1}{\beta - p} \left\{ \beta - 1 - \frac{zf''(z)}{f'(z)} \right\} = \frac{zg'(z)}{g(z)} .$$

Then we have $H(0) = 1$, $\mathrm{Re}\, H(z) > 0$ in D and therefore it follows that $g(z) \in S(1)$.

A simple calculation of (2) gives

$$\frac{f'(z)}{pz^{p-1}} = \left(\frac{g(z)}{z} \right)^{p-\beta}$$

and then we have

$$\frac{zf'(z)}{z^{\frac{p+k}{2}}\, g(z)^{\frac{p-k}{2}}} = p\left(\frac{z}{g(z)}\right)^{\beta - \frac{p+k}{2}}$$

From the result by Robinson [6] and Komatu [1], we have

(3) $$\frac{Zf'(z)}{z^{\frac{p+k}{2}}\, g(z)^{\frac{p-k}{2}}} = p\left(\frac{g(z)}{z}\right)^{\frac{p+k}{2} - \beta} \prec p\left(\frac{1}{1-z}\right)^{p+k-2\beta}$$

where the symbol $\prec$ denotes the subordination.

From the assumption of THEOREM 2 and (3), we have

(4) $$\mathrm{Re}\ \frac{zf'(z)}{z^{\frac{p+k}{2}}\, g(z)^{\frac{p-k}{2}}} > 0 \qquad \text{in D.}$$

On the other hand, let us put

$$G(z) = z^{\frac{p+k}{2}}\, g(z)^{\frac{p-k}{2}}$$

Then we have

$$\mathrm{Re}\ \frac{zG'(z)}{G(z)} = \frac{p+k}{2} + \left(\frac{p-k}{2}\right) \mathrm{Re}\ \frac{zg'(z)}{g(z)}$$

$$= \frac{p+k}{2} + \frac{p-k}{2(\beta-p)}\left\{\beta - 1 - \mathrm{Re}\ \frac{zf''(z)}{f'(z)}\right\}.$$

From the assumption (1), we have

$$\mathrm{Re}\ \frac{zG'(z)}{G(z)} > \frac{p+k}{2} + \frac{(p-k)}{2(\beta-p)} + \frac{(k-p)}{2(\beta-p)}\left\{\beta - (\beta-p)\frac{(k+p)}{(k-p)}\right\}$$

$$= 0 \qquad \text{in D.}$$

On the other hand, G(z) has a zero of order p at z=0 and therefore we have

(5) $$G(z) \in S(p).$$

From (4) and (5), we have

$$f(z) \in K(p).$$

This completes our proof.

COROLLARY 2. Let $f(z) \in A(p)$ and assume that $f(z)$ satisfies one of the following conditions:

$$-p < 1 + \mathrm{Re}\,\frac{zf''(z)}{f'(z)} < p+1 \quad \text{in } D \tag{6}$$

$$-\frac{1}{2}p < 1 + \mathrm{Re}\,\frac{zf''(z)}{f'(z)} < p+\frac{3}{2} \quad \text{in } D. \tag{7}$$

Then $f(z)$ is p-valently close-to-convex in D or $f(z) \in K(p)$.

PROOF. Letting $k=\beta=p+1$ in THEOREM 2, we have (6) and letting $k=p+2$ and $\beta=p+3/2$ in THEOREM 2, we have (7).

Hence our conclusion follows in every case by THEOREM 2.

References

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