数理解析研究所講究録
第 664 巻 1988 年 51-57

A NOTE ON MULTIVALENT FUNCTIONS

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1. INTRODUCTION

Let A_p denote the class of functions of the form

$$f(z) = \sum_{n=p}^{\infty} a_n z^n \qquad (a_p = 1 \ ; \ p \in N = \{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U = \{z:|z| < 1\}$.

A function $f(z)$ belonging the class A_p is said to be p-valently α-convex in the unit disk U if and only if

$$\mathrm{Re}\{(1 - \alpha)\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} + \alpha(1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)})\} > 0 \tag{1.2}$$

for some real α, and for all $z \in U$(cf.[5]).

Denoting by $A_p(\alpha)$ the subclass of A_p consisting of functions which are p-valently α-convex in the unit disk U, we see that $A_p(\alpha)$ is the generalization class of α-convex functions studied by Miller, Mocanu and Reade [2] (or [3],[4]).

Recently, Saitoh, Nunokawa, Owa, Sekine and Fukui [6] have proved some interesting results for functions belonging to the class $A_p(\alpha)$.

2. PROPERTIES OF THE CLASS $A_p(\alpha)$

We begin with the statements of the following lemmas.

LEMMA 1 ([1]). Let $\Phi(u,v)$ be a complex valued function,

$\Phi: D \to C$, $D \subset C \times C$ (C is the complex plane),

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\Phi(u,v)$ satisfies the following conditions:

(i) $\Phi(u,v)$ is continuous in D ;

(ii) $(1,0) \in D$ and $\mathrm{Re}\{\Phi(1,0)\} > 0$;

(iii) $\mathrm{Re}\{\Phi(iu_2,v_1)\} \leqq 0$ for all $(iu_2,v_1) \in D$ and such that

$$v_1 \leqq -(1 + u_2^2)/2.$$

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be regular in the unit disk U such that $(p(z), zp'(z)) \in D$ for all $z \in U$.

If

$$\mathrm{Re}\{\Phi(p(z), zp'(z))\} > 0 \qquad (z \in U) ,$$

then

$$\mathrm{Re}\{p(z)\} > 0 \qquad (z \in U) .$$

LEMMA 2 ([6]). If $f(z) \in A_p(\alpha)$ with $\alpha \geqq 1$, then

$$\mathrm{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \frac{-\alpha + \sqrt{\alpha(\alpha + 8)}}{4} \tag{2.1}$$

for $z \in U$.

PROOF. Define the function $g(z)$ by

$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} = \beta + (1 - \beta)g(z) \tag{2.2}$$

for $f(z) \in A_p(\alpha)$, where

$$\beta(\alpha) = \frac{-\alpha + \sqrt{\alpha(\alpha + 8)}}{4} . \tag{2.3}$$

It follows from the above that $g(z)$ is regular in the unit disk U, and that $g(z) = 1 + g_1 z + g_2 z^2 + \cdots$.

Making the logarithmic differentiations of both sides in (2.2), we have

$$1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} = \beta + (1 - \beta)g(z) + \frac{(1 - \beta)zg'(z)}{\beta + (1 - \beta)g(z)} . \tag{2.4}$$

Thus we can see that

$$\operatorname{Re}\{(1 - \alpha)\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} + \alpha(1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)})\} \tag{2.5}$$

$$= \operatorname{Re}\{\beta + (1 - \beta)g(z) + \frac{\alpha(1 - \beta)zg'(z)}{\beta + (1 - \beta)g(z)}\} > 0$$

for $f(z) \in A_p(\alpha)$. Letting

$$\Phi(u,v) = \beta + (1 - \beta)u + \frac{\alpha(1 - \beta)v}{\beta + (1 - \beta)u} , \tag{2.6}$$

(note that $u = g(z)$ and $v = zg'(z)$), we know that

(i) $\Phi(u,v)$ is continuous in $D = (C - \{\beta/(\beta - 1)\}) \times C$;

(ii) $(1,0) \in D$ and $\operatorname{Re}\{\Phi(1,0)\} = 1 > 0$;

(iii) for all $(iu_2,v_1) \in D$ such that $v_1 \leqq -(1 + u_2^2)/2$,

$$\operatorname{Re}\{\Phi(iu_2,v_1)\} = \beta + \frac{\alpha\beta(1 - \beta)v_1}{\beta^2 + (1 - \beta)^2u_2^2}$$

$$\leqq \beta - \frac{\alpha\beta(1 - \beta)(1 + u_2^2)}{2\{\beta^2 + (1 - \beta)^2u_2^2\}}$$

$$\leqq 0 .$$

Therefore, the function $\Phi(u,v)$ defined by (2.6) satisfies the conditions Lemma 1. It follows from this fact that $\operatorname{Re}\{g(z)\} > 0$, that is, that

$$\operatorname{Re}\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\} > \beta , \tag{2.7}$$

which completes the proof of Lemma 2.

LEMMA 3. Let k denote the real number such that $0 < k < 1$. Then we have the following inequality.

$$\cos k\theta \geqq \cos^k\theta \qquad (\,|\theta| < \pi/2\,)\,. \tag{2.8}$$

PROOF. We put

$$F(\theta) = \cos k\theta - \cos^k\theta \quad .$$

Then we have

$$F'(\theta) = k\left(\frac{\sin\theta}{\cos^{1-k}\theta} - \sin k\theta\right) \geqq 0$$

and $F(0) = 0$.

It follows from the above that

$$F(\theta) \geqq 0 \;.$$

Therefore, we have

$$\cos k\theta \geqq \cos^k\theta \;.$$

Consequently, we complete the proof of Lemma 3.

Applying the above lemmas, we prove

THEOREM. If $f(z)$ $A_p(\alpha)$ with $\alpha \geqq 1$, then

$$\mathrm{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\}^k > \left\{\frac{1}{3 - 2\beta(\alpha)}\right\}^k \qquad (z \in U), \tag{2.9}$$

where $\beta(\alpha)$ is given by (2.3) and $0 < k \leqq 1$ (k ; real number).

PROOF. Step 1. First, we prove for $k = 1$.

Define the function $g(z)$ by

$$\frac{f^{(p-1)}(z)}{z} = \gamma + (1 - \gamma)g(z) \tag{2.10}$$

with

$$\gamma = \frac{1}{3 - 2\beta(\alpha)} \quad . \tag{2.11}$$

Then $g(z) = 1 + g_1 z + g_2 z^2 + \cdots$ is regular in U. Making use of the logarithmic differentiations of both sides in (2.10), we have

$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} = 1 + \frac{(1-\gamma)zg'(z)}{\gamma + (1-\gamma)g(z)} \quad . \tag{2.12}$$

Using Lemma 2, (2.12) leads to

$$\begin{aligned} \text{Re}\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - \beta(\alpha)\} &= \text{Re}\{1 - \beta(\alpha) + \frac{(1-\gamma)zg'(z)}{\gamma + (1-\gamma)g(z)}\} \\ &> 0 \quad . \end{aligned} \tag{2.13}$$

Let $u = u_1 + iu_2$, $v = v_1 + iv_2$, and

$$\Phi(u,v) = 1 - \beta(\alpha) + \frac{(1-\gamma)v}{\gamma + (1-\gamma)u} \tag{2.14}$$

(note that $u = g(z)$ and $v = zg'(z)$). Then, it follows from (2.14) that

(i) $\Phi(u,v)$ is continuous in $D = (C - \{\frac{\gamma}{\gamma - 1}\}) \times C$;

(ii) $(1,0) \in D$ and $\text{Re}\{\Phi(1,0)\} = 1 - \beta(\alpha) > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leqq -(1 + u_2^2)/2$,

$$\begin{aligned} \text{Re}\{\Phi(iu_2,v_1)\} &= 1 - \beta(\alpha) + \frac{\gamma(1-\gamma)v_1}{\gamma^2 + (1-\gamma)^2 u_2^2} \\ &\leqq 1 - \beta(\alpha) - \frac{\gamma(1-\gamma)(1+u_2^2)}{2\{\gamma^2 + (1-\gamma)^2 u_2^2\}} \\ &= \frac{(1-\gamma)(1-2\beta(\alpha))u_2^2}{2\{\gamma^2 + (1-\gamma)^2 u_2^2\}} \\ &\leqq 0 \quad , \end{aligned}$$

because $1 - 2\beta(\alpha) \leqq 0$ for $\alpha \geqq 1$. Thus the function $\Phi(u,v)$ defined by (2.14) satisfies the conditions in Lemma 1. THis implies that $\text{Re}\{g(z)\} > 0$ $(z \in U)$, that is, that

$$\text{Re}\{\frac{f^{(p-1)}(z)}{z}\} > \gamma \quad . \tag{2.15}$$

Therefore, we complete the assertion for $k = 1$.

Step 2. In the next place, we prove for $0 < k < 1$.

Letting

(2.16) $$\frac{f^{(p-1)}(z)}{z} = h(z)$$

and

(2.17) $$\gamma = \frac{1}{3 - 2\beta(\alpha)} > 0 \ .$$

In Step 1, we have

(2.18) $$\mathrm{Re}\{h(z)\} > \gamma > 0 \ .$$

Now, we put

$$h(z) = \rho(\cos\theta + i\sin\theta) \ .$$

From (2.18), we can see that

(2.19) $$\rho\cos\theta > \gamma > 0 \qquad (\ |\theta| < \pi/2 \ ) \ .$$

Therefore,

$$\begin{aligned} \mathrm{Re}\{h(z)\}^k &= \mathrm{Re}\{\rho(\cos\theta + i\sin\theta)\}^k \\ &= \mathrm{Re}\ \rho^k(\cos k\theta + i\sin k\theta) \\ &= \rho^k \cos k\theta \\ &\geqq \rho^k \cos^k\theta \qquad (\text{ by Lemma 3 }) \\ &= (\rho\cdot\cos\theta)^k \\ &> \gamma^k \ . \end{aligned}$$

Hence, we complete the assertion for $0 < k < 1$.

Accordingly, we complete the proof of Theorem.

Making $\alpha = 1$ and $p = 1$ in Theorem, we have

COROLLARY. If the function $f(z) = z + a_2z^2 + \cdots$ is convex in U, then

(2.20) $$\mathrm{Re}\left\{\frac{f(z)}{z}\right\}^{k} > \left(\frac{1}{2}\right)^{k} \qquad (z \in U)$$

for all real number k $(0 < k \leq 1)$.

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