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2-MULTIGRAPHS

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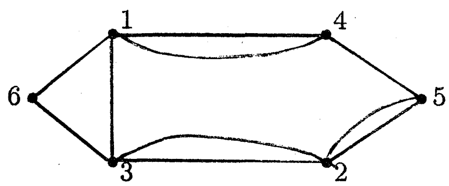
This lecture is based on the two papers [1] and [2] written jointly with T.S. Michael. We refer to these articles for the references not provided here.

Let $G = (V,E)$ be a <u>multigraph</u>. Thus $V = \{1,2,\ldots,n\}$ is a set of n <u>vertices</u> and E is a multiset of unordered pairs of distinct vertices called <u>edges</u>. In a 2-<u>multigraph</u> each edge occurs at most twice in E, equivalently at most two edges join each pair of distinct vertices.

(1) <u>Example</u>



Let d_i equal the number of edges that meet vertex i. Then $d_1 = 4$, $d_2 = 4$, $d_3 = 4$, $d_4 = 3$, $d_5 = 3$, and $d_6 = 2$. The <u>degree sequence</u> is $(4,4,4,3,3,2)$.

With proper labelling of the vertices we may always assume that the degree sequence $D = (d_1,\ldots,d_n)$ of a multigraph satisfies $d_1 \geq \cdots \geq d_n$. This assumption is made implicitly throughout.

Our object of study is the class $\mathfrak{G}_2(D)$ of all 2-multigraphs with the same degree sequence $D = (d_1,\ldots,d_n)$. The first question that arises is that of the nonemptiness of this class.

(2) Theorem (Chungphaisan 1974). There exists a 2-multigraph with degree sequence $D=(d_1,\ldots,d_n)$ if and only if

(3) $d_1 + \cdots + d_n$ is even ,

(4) $$\sum_{i=1}^{k} d_i \leq 2k(k-1) + \sum_{i=k+1}^{n} \min\{2k,d_i\} \quad (k = 1,\ldots,n) .$$

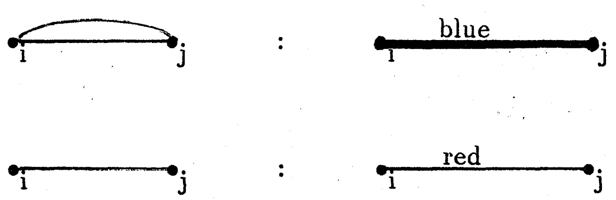
The above is a special case of a more general theorem of Chungphaisan for r-multigraphs obtained by replacing the three 2's above by r's. The case $r=1$ is then the following well know result.

(5) Theorem (Erdös/Gallai 1960). There exists a graph with degree sequence $D=(d_1,\ldots,d_n)$ if and only if

(6) $d_1 + \cdots + d_n$ is even ,

(7) $$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k,d_i\} \quad (k = 1,\ldots,n) .$$

We now adopt the following color convention for 2-multigraphs. An edge of multiplicity 2 is denoted by one blue edge (designated by a bold edge). An edge of multiplicity 1 is denoted by one red edge:



Thus a 2-multigraph is a graph in which every edge is colored red or blue, where red counts as 1 and blue counts as 2.

(8) <u>Example</u>. Two 2-multigraphs with the degree sequence $D = (4,4,4,4,4,)$:

The example on the right is parsimonious in the sense that it has the smallest number of colored edges among all 2-multigraphs in $\mathfrak{G}_2(D)$.

Let G be a 2-multigraph in $\mathfrak{G}_2(D)$. Then

$\beta(G)$ = number of blue edges of G,

$\rho(G)$ = number of red edges of G,

$\tau(G)$ = number of colored edges of G,

m(G) = number of edges of G counting multiplicites.

It follows that

(9) $\tau(G) = \beta(G) + \rho(G),$

(10) $$\begin{aligned} m(G) &= (d_1 + \cdots + d_n)/2 \\ &= 2\beta(G) + \rho(G) \\ &= \tau(G) + \beta(G) \\ &= 2\tau(G) - \rho(G)\ . \end{aligned}$$

The 2-multigraph G is called *parsimonious* provided $\tau(G) \leq \tau(H)$ for all 2-multigraphs H with the same degree sequence D as G. It follows from (10) that each of the following is equivalent to the parsimony of G:

$$\beta(G) \geq \beta(H) \ \text{ for all } \ H \in \mathfrak{G}_2(D) \ ,$$

$$\rho(G) \leq \rho(H) \ \text{ for all } \ H \in \mathfrak{G}_2(D) \ .$$

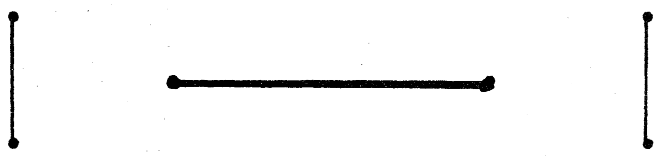
(11) *Example*. The following two 2-multigraphs with degree sequence (3,3,2,1,1) are parsimonious:

The *red graph* of a 2-multigraph is the graph (not the 2-multigraph) determined by its red edges with isolated vertices deleted.

(12) *Theorem.* *Each connected component of the red graph of a parsimonious 2-multigraph is either a star (a $K_{1,t}$ for some $t \geq 1$) or a triangle (a K_3). Furthermore at most one connected component is a triangle.*

(13) *Example*. The converse of Theorem (12) does not hold. The 2-multigraph

with degree sequence $D = (2,2,1,1,1,1)$ has a red graph consisting of two stars $K_{1,2}$, but the 2-multigraph with degree sequence D



has fewer colored edges.

A galaxy is a graph each of whose connected components is a star or a triangle with at most one component equal to a triangle. A stellar galaxy is a galaxy with no triangle. By Theorem (12) the red graph of a parsimonious 2-multigraph is a galaxy.

(14) Remark. The Erdös/Gallai Theorem gives necessary and sufficient conditions that the parsimonious 2-multigraphs in a class $\mathfrak{G}_2(D)$ have empty red graphs (i.e. all blue edges). These conditions are

(15) d_i is even $(i = 1,\dots,n)$,

(16) $d_1 + \cdots + d_n \equiv 0 \pmod 4$,

(17) $$\sum_{i=1}^{k} d_i \le 2k(k-1) + \sum_{i=k+1}^{n} \min\{2k,d_i\} \quad (k = 1,\dots,n)\,.$$

(The d_i clearly have to be even. Now apply the Erdös/Gallai Theorem to the degree sequence $(d_1/2,\dots,d_n/2)$ and color all edges of the resulting graph blue.) Condition (17) is the Chungphaisan condition (4). If we retain the condition (15), the only other possibility for $d_1 + \cdots + d_n$ is

(18) $$d_1 + \cdots + d_n \equiv 2 \pmod 4.$$

Since the d_i are even, the red graph of a parsimonious 2-multigraph cannot contain any stars. It follows that (15), (17), and (18) are necessary and sufficient conditions in order that the parsimonious 2-multigraphs in $\mathfrak{G}_2(D)$ have red graphs consisting only of a red triangle.

For the two cases discussed in Remark (14) there are algorithms to construct a parsimonious 2-multigraph in a nonempty class $\mathfrak{G}_2(D)$. In the case that (15), (16), and (17) are satisfied it suffices to apply the well-known algorithm of Havel and Hakimi to the degree sequence $(d_1/2,\dots,d_n/2)$ and then to color all edges of the resulting graph blue. In the case that (15), (17), and (18) are satisfied the algorithm of Havel and Hakimi can be refined to produce a parsimonious 2-multigraph (the problem is to decide where the red edges of the triangle go).

(19) Problem. Find an algorithm to construct a parsimonious 2-multigraph in a nonempty class $\mathfrak{G}_2(D)$.

We now consider the question: What kinds of red graphs [galaxies] can occur in the parsimonious 2-multigraphs in one class $\mathfrak{G}_2(D)$?

A collection $\mathcal{G}$ of galaxies is class-compatible provided there is a class $\mathfrak{G}_2(D)$ such that for each $H \in \mathcal{G}$ there is a parsimonious 2-multigraph $G_H \in \mathfrak{G}_2(D)$ whose red graph is isomorphic to H.

(20) Remark. The galaxies of a class-compatible collection $\mathcal{G}$ of galaxies contain

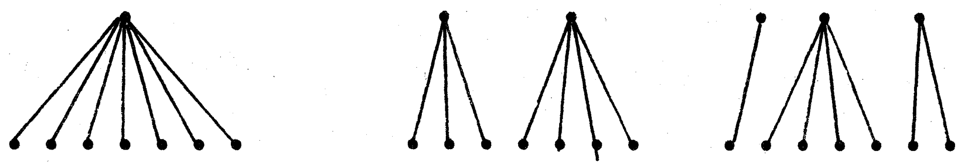
(21) The same number ρ of edges (by parsimony),

(22) The same number μ of vertices of odd degree (the number of odd components of the degree sequence D).

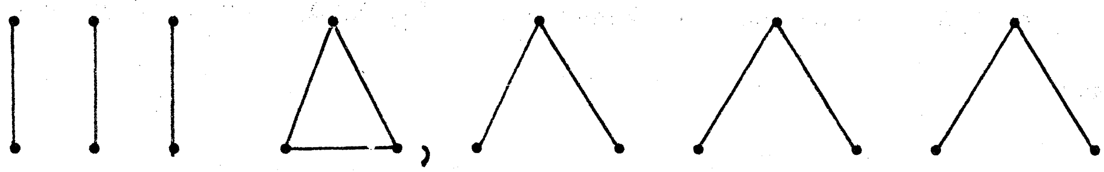
If we denote by Δ $(= 0$ or $1)$ the number of triangles in a galaxy, then the number of odd stars [stars whose center has odd degree] equals $\mu-\rho+3\Delta$. But we do not know whether the galaxies in a class-compatible collection all have the same value of Δ (thus we do not know whether they have the same number of odd stars). We do however have the following result.

(23) Theorem. Let $\mathcal{G}$ be a collection of stellar galaxies having the same number ρ of edges and the same number μ of vertices of odd degree (equivalently the same number of odd stars). Then $\mathcal{G}$ is class-compatible.

(24) Example. The three galaxies below (with $\rho=7$ and $\mu=8$) are class-compatible:

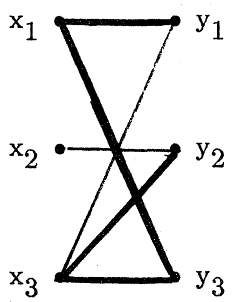


(25) Question. Are the following two galaxies (one of which has a triangle and the other of which does not) class compatible?



The preceding theory has an analogue for bipartite 2-multigraphs with prescribed degree sequence. The vertices of a bipartite 2-multigraph are partitioned into two sets X and Y with each edge joining vertices from different sets. We let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be the degree sequences of the vertices in the two parts. Let $\mathfrak{B}_2(R,S)$ denote the class of bipartite 2-multigraphs with these degree sequences R and S. The class $\mathfrak{B}_2(R,S)$ corresponds to the class $\mathfrak{A}_2(R,S)$ of all (0,1,2)-matrices with row sum vector R and column sum vector S (use the reduced adjacency matrix).

(26) Example.



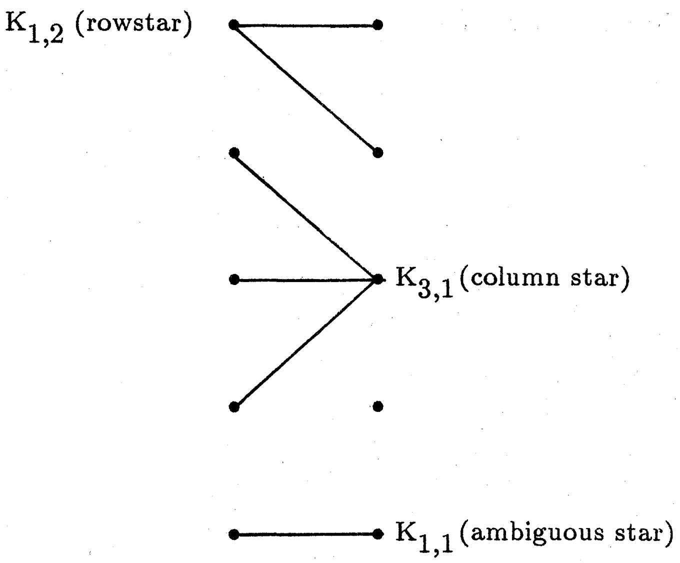
$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \left[\begin{array}{ccc} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{array}\right] \end{array}$$

$$R = (4,1,5), \qquad S = (3,3,4)$$

The notion of parsimony carries over to the class $\mathfrak{B}_2(R,S)$. In terms of $\mathfrak{A}_2(R,S)$ we obtain that a matrix A in $\mathfrak{A}_2(R,S)$ is parsimonious provided it has the largest number of 0's (equivalently the smallest number of 1's, equivalently the largest number of 2's) among all matrices in $\mathfrak{A}_2(R,S)$.

Arguments analogous to those for $\mathfrak{G}_2(D)$ allow one to conclude that the red graphs of parsimonious bipartite 2-multigraphs have their connected components equal to stars. Because of the bipartite assumption there can be no triangle, however a different difficulty arises. Some stars $K_{1,t} (t > 1)$ have their center in X (corresponding in the matrix formulation to 1's in the same row) while other stars $K_{t,1}$ $(t > 1)$ have their center in Y (corresponding to 1's in the same column). The former are called <u>row stars</u> while the latter are called <u>column stars</u>. Then there are the <u>ambiguous</u> (or unoriented) stars $K_{1,1}$ (corresponding to a 1 which is the only 1 in its row <u>and</u> the only 1 in its column.

(27) <u>Example</u>.



$$\begin{bmatrix} 1 & 1 & & \\ & & 1 & \\ & & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

We use the concepts of a galaxy and of class-compatibility of galaxies in an analogous way to that used above. If a collection of galaxies is class-compatible then they have

(28) the same number ρ of 1's (edges),

(29) the same number λ of rows with an odd number of 1's,

(30) the same number ν of columns with an odd number of 1's .

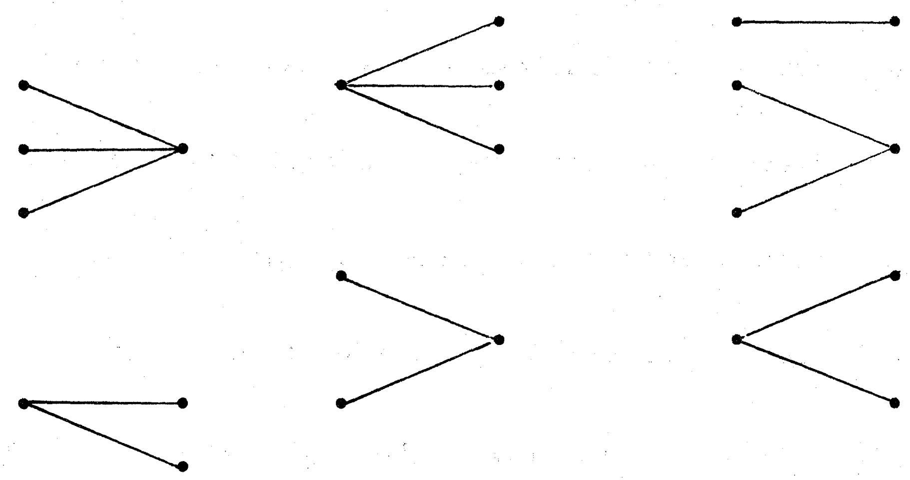
In addition since $\lambda+\nu-\rho$ equals the number of odd stars (includes ambiguous stars), they have

(31) the same number λ of odd stars.

(32) Theorem. A collection of glaxies with the same ρ, λ, and ν is class-compatible provided it is possible to assign an orientation (row or column) to each ambiguous star of each galaxy so that each galaxy has the same number of odd row stars and the same number of odd column stars.

The preceding theorem may be true without the orientation assumption.

(33) Example. The following three galaxies have $\rho=5$, $\lambda=3$ and $\nu=3$ but do not satisfy the orientation assumption in Theorem (3.2).



1 odd column star — 1 odd row star — 1 odd star (row or column?)

Are these galaxies class-compatible?

REFERENCES

1. Richard A. Brualdi and T. S. Michael, The class of 2-multigraphs with a prescribed degree sequence, Linear and Multilinear Algebra, to appear.

2. Richard A. Brualdi and T. S. Michael, The class of matrices of 0's, 1's, and 2's with prescribed row and column sums, Linear Algebra and its Applications, to appear.