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A Criterion for Finding Existence and Nonexistence Domains of Solutions of Nonlinear Equations

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Let X and Y be Banach spaces, f and g be operators on $D\subsetneq X$ with values in Y, where f is Fréchet differentiable in an open convex set $D_0\subset D$, while the differentiability of g is not assumed. Let $B(x^0,r)$ be the open ball with center x^0 and radius r in X, $\bar{B}(x^0,r)$ denote its closure and $\bar{B}(x^0,R)\subset D_0$.

To find a solution x^* of the equation

$$f(x)+g(x)=0, \tag{1}$$

several authors[6,7,9-11] have considered the iteration

$$x_0=x^0, \quad x_{n+1}=x_n-f'(x_n)^{-1}(f(x_n)+g(x_n)),\ n\geqq 0.$$

In this paper, we consider the iteration

$$x_0\in\bar{B}(x^0,R), \quad x_{n+1}=x_n-A(x_n)^{-1}(f(x_n)+g(x_n)), \quad n\geqq 0, \tag{2}$$

where $A(x)$ is an approximation for $f'(x)$. Assume that $A(x^0)^{-1}$ exists and that for any $x,y\in\bar{B}(x^0,r)\subseteq\bar{B}(x^0,R)$, the following hold:

$$\|A(x^0)^{-1}(A(x)-A(x^0))\| \leqq w_0(\|x-x^0\|)+b,$$

$$\|A(x^0)^{-1}(f'(x+t(y-x))-A(x))\|$$
$$\leqq w(\|x-x^0\|+t\|y-x\|)-w_0(\|x-x^0\|)+c, \qquad t\in[0,1],$$

$$\|A(x^0)^{-1}(g(x)-g(y))\| \leqq e(r)\|x-y\|,$$

where $w(r+\tau)-w_0(r), \tau\geqq 0$ and $e(r)$ are nondecreasing functions with $w(0)=w_0(0)=e(0)=0$, $w_0(r)$ is differentiable $w_0'(r)>0$ at every point of $[0,R]$, and the constants b,c satisfy $b\geqq 0$, $c\geqq 0$ and $b+c<1$. Put

$$\eta=\|A(x^0)^{-1}(f(x^0)+g(x^0))\|>0 , \qquad \phi(r)=\eta-r+\int_0^r w(t)dt,$$

$$\psi(r)=\int_0^r e(t)dt, \qquad \chi(r)=\phi(r)+\psi(r)+(b+c)r,$$

and denote by χ^* and r^* the minimal value and the minimal point of $\chi(r)$ in $[0,R]$. Furthermore, let

$$\omega(r)=1-w_0(r)-b, \qquad r_x=\|x-x^0\|,$$

$$\eta_x=\|A(x)^{-1}(f(x)+g(x))\|, \qquad v_x(r)=\omega(r_x)\eta_x-\chi(r_x)+\chi(r).$$

Theorem 1. For some $x_0\in\bar{B}(x^0,R)$, the iteration (2) is well defined for all $n\geqq 0$, and $\{x_n\}$ converges to a solution $x^*\in B(x^0,r^*)$ of (1) if and only if there exists an $x\in B(x^0,r^*)$ such that $v_x(r^*)<0$.

Theorem 2. Let $x \in B(x^0, r^*)$.

(i) If $v_x(r^*)>0$, then there is no solution of (1) in $B(x,\delta)\cap\bar{B}(x^0,R)$, where δ is a unique positive root of the scalar equation

$$q(t)=\omega(r_x)^{-1}(\chi(t+r_x)-\chi(r_x))+2t-\eta_x=0.$$

We have $\delta \geqq \min\{r^*-r_x,\ a_x/2\}$ and $a_x > (\chi(r_x)-\chi(r^*))/\omega(r_x)$.

(ii) If $v_x(r^*)\leqq 0$, then there exists a unique solution x^* of (1) in $\bar{B}(x,r^*-r_x)\subseteq\bar{B}(x^0,r^*)$, which can be obtained by (2) with $x_0=x$.

For the case of Newton's method applied to the equation

$$f(x)=0, \tag{3}$$

we assume that $f'(x^0)^{-1}$ exists and $f'(x^0)^{-1}f'(x)$ satisfies a Lipschitz condition in D_0 with a Lipschitz constant $K>0$ and $1/K\leqq R$. Then we can take $A(x)=f'(x)$, $g(x)=0$, $w_0(r)=w(r)=Kr$, $b=c=e(r)=0$, so that we have $\chi(r)=\eta-r+Kr^2/2$. Hence Theorems 1 and 2 reduce to the following:

Corollary 1. For some $x_0\in\bar{B}(x^0,R)$, Newton's method is well defined for all $n\geqq 0$ and $\{x_n\}$ converges to a solution $x^*\in B(x^0,1/K)$ of (3), if and only if there is an $x\in B(x^0,1/K)$ such that $h_x=Ka_x/(1-Kr_x)<1/2$.

Corollary 2. Let $x \in B(x^o, 1/K)$.

(i) If $h>1/2$, then there is no solution of (3) in $B(x,\delta)\cap\bar{B}(x^o,R)$, where $\delta=2a_x/(1+\sqrt{1+2h_x})\geq\min\{r^*-r_x, a_x/2\}$, and $a_x>(1-Kr_x)/(2K)$.

(ii) If $h_x\leq 1/2$, then there exists a unique solution of (3) in $B(x,1/K-r_x)\subseteq\bar{B}(x^o,1/K)$, which can be obtained by Newton's method starting from x.

Theorems 1 and 2 generalize and deepen Rheinboldt-Dennis's results for Newton-like methods. Furthermore, Theorem 2 gives a foundation for constructing an algorithm which finds all the solutions of the equation (1) in a domain.

References

1). X. Chen and T. Yamamoto, Convergence domains of certain iterative methods for solving nonlinear equations, preprint;

2). J. E. Dennis, On the convergence of Newton-like methods, in Numerical Methods for Nonlinear Aigebraic Equations, ed. P.Rabinowitz, Gordon and Breach, New York(1970), 163-181;

3). W. B. Gragg and R. A. Tapia, Optimal error bounds for the Newton-Kantorovich theorem, SIAM J.Numer. Anal. 11(1974), 10-13;

4). L.V.Kantorovich, On Newton's method for functional equations, Dokl.Akad.Nauk SSSR59(1948),1237-1240;

5). J. M. Ortega and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York(1970);

6). W.C. Rheinboldt, A unified convergence theory for a class of iterative process, SIAM J. Numer. Anal. 5(1968),42-63;

7). T. Yamamoto, Error bounds for Newton-like methods under Kantorovich type assumptions, Japan J. Appl. Math. 3(1986),295-313;

8). T. Yamamoto, A convergence theorem for Newton-like methods in Banach spaces,Numer.Math.51(1987),545-557;

9). T. Yamamoto, A note on a posteriori error bound of Zabrejko and Nguen for Zincenko's iteration, Numer. Funct. Anal. and Optimiz.9(1987),987-994;

10). P. P. Zabrejko and D. F. Nguen, The majorant method in the theory of Newton-Kantorovich approximations and the Pták error estimates, Numer. Funct. Anal. and Optimiz. 9(1987),671-684;

11). A. I. Zincenko, Some approximate methods of solving equations with nondifferentiable operators Dopovidi Akad. Nauk Ukrain. RSR(1963),156-161.