数理解析研究所講究録
第 674 巻 1988 年 127-133

On the Norm of Block Products of Matrices

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1. Introduction and Preliminaries

Let $M_{m,n}$ be the space of all $m \times n$ complex matrices, and set $M_n = M_{n,n}$. For each $A \in M_{m,n}$ the vector of singular values of A (i.e. eigenvalues of $|A| = (A^*A)^{1/2} \in M_n$) arranged in decreasing order is denoted by

$$\sigma(A) = (\sigma_1(A), \sigma_2(A), \cdots, \sigma_n(A)).$$

For $1 \leq p < \infty$, we denote the p-norm of A by $||A||_p$, i.e.

$$||A||_p = \left[\mathrm{tr}(|A|^p)\right]^{1/p} = \left[\sum_{i=1}^{n} \sigma_i(A)^p\right]^{1/p},$$

and the spectral norm (or operator norm) by $||A||_\infty = \sigma_1(A)$.

It is well-known that for $A, B \in M_n$ the following Hölder-type norm inequality holds:

$$||AB||_r \leq ||A||_p ||B||_q \qquad \text{whenever} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{1}$$

This can be implied from the inequalities

$$\sum_{i=1}^{k} \sigma_i(AB) \leq \sum_{i=1}^{k} \sigma_i(A)\sigma_i(B) \qquad \text{for} \quad k = 1, 2, \cdots, n. \tag{2}$$

Furthermore, stronger inequalities hold:

$$\prod_{i=1}^{k} \sigma_i(AB) \leq \prod_{i=1}^{k} \sigma_i(A)\sigma_i(B) \qquad \text{for} \quad k = 1, 2, \cdots, n. \tag{3}$$

For $A = [a_{ij}], B = [b_{ij}] \in M_n$, their Schur product (or Hadamard product) $A \circ B$ is defined by the entrywise multiplication

$$A \circ B = [a_{ij} b_{ij}]_{i,j=1}^{n}.$$

Recently it has shown that the following similar inequalities hold ([3], [5]):

$$\sum_{i=1}^{k} \sigma_i(A \circ B) \leq \sum_{i=1}^{k} \sigma_i(A)\sigma_i(B) \qquad \text{for} \quad k = 1, 2, \cdots, n. \tag{4}$$

These imply the Hölder-type norm inequality

$$||A \circ B||_r \leq ||A||_p ||B||_q \qquad \text{whenever} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{5}$$

(See [1], [2] and [6] for related results.)

In the present article, we are interested in the problem to find a product (of two matrices) which unifies the ordinary matrix product and the Schur product and satisfies the Hölder-type norm inequalities. There are two quite natural candidates called box products: let $A, B \in M_n$ be partitioned into N^2 blocks; $A = [A_{ij}]_{i,j=1}^N$, $B = [B_{ij}]_{i,j=1}^N$ with $A_{ij}, B_{ij} \in M_p$ $(n = Np)$. We define block products $A \square B$ and $A \blacksquare B$ by

$$A \square B = [A_{ij} B_{ij}]_{i,j=1}^N \qquad \text{and} \qquad A \blacksquare B = [\textstyle\sum_{k=1}^N A_{ik} \circ B_{kj}]_{i,j=1}^N.$$

If we consider the trivial partition $N = n, p = 1$, then $A \square B = A \circ B$ and $A \blacksquare B = AB$, while if $N = 1, p = n$, then $A \square B = AB$ and $A \blacksquare B = A \circ B$. We investigate these products in the next section.

For later use, we explain a notion and elementary facts of majorization. Let $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$ be vectors in $\mathbb{R}^n$. We denote the decreasing rearrangements of the components of ξ by $\xi_{[1]} \geq \xi_{[2]} \geq \cdots \geq \xi_{[n]}$. ξ is said to be submajorized by η (in symbols $\xi \prec_w \eta$) if

$$\sum_{i=1}^k \xi_{[i]} \leq \sum_{i=1}^k \eta_{[i]} \qquad \text{for} \quad k = 1, 2, \cdots, n.$$

If in addition $\sum_{i=1}^n \xi_i = \sum_{i=1}^n \eta_i$ holds, then ξ is said to be majorized by η (in symbols $\xi \prec \eta$). Inequalities (2) and (4) can be expressed by submajorization

$$\sigma(AB) \prec_w \sigma(A) \cdot \sigma(B) \qquad \text{and} \qquad \sigma(A \circ B) \prec_w \sigma(A) \cdot \sigma(B),$$

where we denotes the coordinatewise product of vectors $\sigma(A)$ and $\sigma(B)$ by $\sigma(A) \cdot \sigma(B)$. Submajorization for the sum of matrices is also known:

$$\sigma(A + B) \prec_w \sigma(A) + \sigma(B). \tag{6}$$

It is a basic fact that submajorization is preserved by the increasing convex functions: if $\xi \prec_w \eta$, then $f(\xi) \prec_w f(\eta)$ for all increasing convex function f, where $f(\xi)$ denotes the vector $(f(\xi_1), f(\xi_2), \cdots, f(\xi_n))$. In particular, if $\xi, \eta \in \mathbb{R}_+^n$ and

$$\prod_{i=1}^k \xi_{[i]} \leq \prod_{i=1}^k \eta_{[i]} \qquad \text{for} \quad k = 1, 2, \cdots, n,$$

then $\xi \prec_w \eta$. See [4] for further details.

2. Results

First we consider the box product $A \,\square\, B$.

Lemma 1. *For any $A, B \in M_n$*

$$\begin{bmatrix} \mathcal{E}(B^*B) & (A \,\square\, B)^* \\ A \,\square\, B & \mathcal{E}(AA^*) \end{bmatrix} \geq 0, \tag{7}$$

where $\mathcal{E} : M_n \to M_n$ denotes the pinching, i.e.

$$\mathcal{E}(X) = [\delta_{ij} X_{ij}]_{i,j=1}^N \qquad \text{for} \quad X = [X_{ij}]_{i,j=1}^N \in M_n.$$

Proof. Take any vectors $\xi = [\xi_j]_{j=1}^N$, $\eta = [\eta_j]_{j=1}^N \in \mathbb{C}^n$ with $\xi_j, \eta_j \in \mathbb{C}^p$. Then

$$\begin{aligned}
\left|< (A \,\square\, B)\xi | \eta >\right|^2 &= \Big|\sum_{i,j=1}^N < A_{ij} B_{ij} \xi_j | \eta_i >\Big|^2 \\
&= \Big|\sum_{i,j=1}^N < B_{ij} \xi_j | A_{ij}^* \eta_i >\Big|^2 \\
&\leq \Big\{\sum_{i,j=1}^N ||B_{ij}\xi_j|| \cdot ||A_{ij}^* \eta_i||\Big\}^2 \\
&\leq \Big\{\sum_{i,j=1}^N ||B_{ij}\xi_j||^2\Big\} \cdot \Big\{\sum_{i,j=1}^N ||A_{ij}^* \eta_i||^2\Big\} \\
&= \Big\{\sum_{j=1}^N < (\sum_{i=1}^N B_{ij}^* B_{ij})\xi_j | \xi_j >\Big\} \cdot \Big\{\sum_{i=1}^N < (\sum_{j=1}^N A_{ij} A_{ij}^*)\eta_i | \eta_i >\Big\} \\
&= < \mathcal{E}(B^*B)\xi | \xi > \cdot < \mathcal{E}(AA^*)\eta | \eta >,
\end{aligned}$$

which shows that (7) holds. ∎

Using this lemma we have the following.

Theorem 2. *For any $A, B \in M_n$*

$$\sum_{j=1}^k \sigma_j (A \,\square\, B)^2 \leq \sum_{j=1}^k \sigma_j(A)^2 \sigma_j(B)^2 \qquad \text{for} \quad k = 1, 2, \cdots, n. \tag{8}$$

Proof. By (7) there is $C \in M_n$ such that $\|C\|_\infty \le 1$ and

$$A \circ B = \mathcal{E}(AA^*)^{1/2} \cdot C \cdot \mathcal{E}(B^*B)^{1/2}.$$

By (3) this implies

$$\prod_{j=1}^{k} \sigma_j(A \circ B)^2 \le \prod_{j=1}^{k} \sigma_j(\mathcal{E}(AA^*))\sigma_j(\mathcal{E}(B^*B)) \qquad \text{for} \quad k = 1, 2, \cdots, n,$$

and consequently

$$\sum_{j=1}^{k} \sigma_j(A \circ B)^2 \le \sum_{j=1}^{k} \sigma_j(\mathcal{E}(AA^*))\sigma_j(\mathcal{E}(B^*B)) \qquad \text{for} \quad k = 1, 2, \cdots, n.$$

Let ω be a primitive N th root of 1, and define the unitary matrix $U = [\delta_{ij}\omega^j I_p]_{i,j=1}^N \in M_n$. Since the pinching $\mathcal{E}$ can be written in the form

$$\mathcal{E}(X) = \frac{1}{N}\sum_{k=1}^{N} U^{*k} X U^k \qquad \text{for} \quad X \in M_n, \tag{9}$$

we get by (6)

$$\sum_{j=1}^{k} \sigma_j(\mathcal{E}(AA^*)) \le \sum_{j=1}^{k} \sigma_j(AA^*) = \sum_{j=1}^{k} \sigma_j(A)^2$$

and

$$\sum_{j=1}^{k} \sigma_j(\mathcal{E}(B^*B)) \le \sum_{j=1}^{k} \sigma_j(B^*B) = \sum_{j=1}^{k} \sigma_j(B)^2.$$

Hence, by elementary calculation, we have (8). ∎

As the consequence of the last theorem we have the norm inequalities.

Corollary 3. *Whenever* $p, q, r \ge 2$ *satisfy* $1/r = 1/p + 1/q$,

$$\|A \circ B\|_r \le \|A\|_p \|B\|_q. \tag{10}$$

In particular

$$\|A \circ B\|_\infty \le \|A\|_\infty \|B\|_\infty. \tag{11}$$

Note that Lemma 1 and norm inequality (11) remain valid in the C^*-algebra setting. In fact, we can obtain

$$\left\|[A_{ij}B_{ij}]_{i,j=1}^N\right\| \le \left\|[A_{ij}]_{i,j=1}^N\right\| \cdot \left\|[B_{ij}]_{i,j=1}^N\right\|, \tag{12}$$

where $A = [A_{ij}]_{i,j=1}^N$, $B = [B_{ij}]_{i,j=1}^N \in M_n(\mathcal{A})$ with a C^*-algebra $\mathcal{A}$.

Next we consider the box product $A \blacksquare B$. Let $\{e_i\}_{i=1}^n$ be the cannonical basis of $\mathbb{C}^n$, and define the unitary matrix $V \in M_n$ by

$$Ve_{N(k-1)+j} = e_{p(j-1)+k} \qquad \text{for} \quad j = 1, 2, \cdots, N, \; k = 1, 2, \cdots, p.$$

For $A, B \in M_n$, let $C = V^*AV$, $D = V^*BV$. Then we have

$$A \blacksquare B = V(C \square D)V^*, \tag{13}$$

where the block product $\square$ in the right hand side is the one with respect to the partition into p^2 blocks; $C = [C_{k\ell}]_{k,\ell=1}^p$, $D = [D_{k\ell}]_{k,\ell=1}^p$ with $C_{k\ell}, D_{k\ell} \in M_N$.

The next theorem follows from (13) and Theorem 2.

Theorem 4. *For any $A, B \in M_n$*

$$\sum_{j=1}^k \sigma_j(A \blacksquare B)^2 \leq \sum_{j=1}^k \sigma_j(A)^2 \sigma_j(B)^2 \qquad \text{for} \quad k = 1, 2, \cdots, n. \tag{14}$$

The following is a consequence of this theorem.

Corollary 5. *Whenever $p, q, r \geq 2$ satisfy $1/r = 1/p + 1/q$,*

$$||A \blacksquare B||_r \leq ||A||_p ||B||_q. \tag{15}$$

In particular

$$||A \blacksquare B||_\infty \leq ||A||_\infty ||B||_\infty. \tag{16}$$

Finally we remark that there is another approach to the norm inequalities of the box products. The idea is the following: let $\Phi(\cdot, \cdot)$ be a bilinear map from $M_n \times M_n$ to M_n. If there are linear maps Φ_ℓ from M_n to $M_{n,m}$ and Φ_r from M_n to $M_{m,n}$ (for some m) satisfying

$$\begin{aligned} \Phi(A, B) &= \Phi_\ell(A)\Phi_r(B), \\ ||\Phi_\ell(A)||_\infty \leq ||A||_\infty \quad &\text{and} \quad ||\Phi_r(B)||_\infty \leq ||B||_\infty, \end{aligned} \tag{16}$$

for any $A, B \in M_n$, then

$$||\Phi(A, B)||_\infty \leq ||A||_\infty ||B||_\infty.$$

When we consider the bilinear map $\Phi(A, B) = A \,\square\, B$, we can find nice maps Φ_ℓ and Φ_r: for $A = [A_{ij}]_{i,j=1}^N$ and $B = [B_{ij}]_{i,j=1}^N$ define

$$\Phi_\ell(A) = \left[\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_n\right], \qquad \Phi_r(B) = \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \\ \vdots \\ \widehat{B}_n \end{bmatrix},$$

where

$$\widetilde{A}_k = \left[\delta_{ij} A_{ik}\right]_{i,j=1}^N, \quad \widehat{B}_k = \left[\delta_{kj} B_{ij}\right]_{i,j=1}^N \in M_n \qquad \text{for} \quad k = 1, 2, \cdots, n.$$

Then we can check that Φ_ℓ and Φ_r satisfy (16). This nice idea was discovered by P. Nylen.

3. Counterexample

For the box products, desired inequalities are the following:

$$\sum_{j=1}^{k} \sigma_j(A \,\square\, B) \leq \sum_{j=1}^{k} \sigma_j(A)\sigma_j(B) \qquad \text{for} \quad k = 1, 2, \cdots, n. \tag{17}$$

Though inequalities (8) hold, (17) or even the weaker inequalities

$$\sum_{j=1}^{k} \sigma_j(A \,\square\, B) \leq \{\sum_{j=1}^{k} \sigma_j(A)\} \cdot \sigma_1(B) \qquad \text{for} \quad k = 1, 2, \cdots, n \tag{18}$$

do not hold. A counterexample is the following: taking the 4×4 matrices

$$A = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad B = \begin{bmatrix} E_{11} & E_{21} \\ E_{12} & E_{22} \end{bmatrix},$$

where E_{ij} is 2×2 matrix whose (i, j)-entry is equal to 1 and all other entries are 0, we can easily compute the block product

$$A \,\square\, B = \begin{bmatrix} E_{11} & E_{11} \\ E_{22} & E_{22} \end{bmatrix}.$$

Hence we have

$$\sigma(A) = \{2, 0, 0, 0\},$$
$$\sigma(B) = \{1, 1, 1, 1\},$$
$$\sigma(A \,\square\, B) = \{\sqrt{2}, \sqrt{2}, 0, 0\},$$

which do not satisfy (18). In view of (13) the inequalities

$$\sum_{j=1}^{k} \sigma_j(A \blacksquare B) \le \sum_{j=1}^{k} \sigma_j(A)\sigma_j(B) \qquad \text{for} \quad k = 1, 2, \cdots, n \tag{19}$$

or even the weaker inequalities

$$\sum_{j=1}^{k} \sigma_j(A \blacksquare B) \le \{\sum_{j=1}^{k} \sigma_j(A)\} \cdot \sigma_1(B) \qquad \text{for} \quad k = 1, 2, \cdots, n \tag{20}$$

do not hold.

Finally the box products do not meet our request. Our purpose does not have been attained. But we do not have another candidate.

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