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Analytic Subalgebras Associated with Integrable Flows on von Neumann Algebras

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1. Introduction

Let M be a von Neumann algebra and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a σ-weakly continuous flow on M; i.e. suppose that $\{\alpha_t\}_{t\in\mathbb{R}}$ is a one-parameter group of *-automorphisms of M and that for each ρ in the predual, M_*, of M and for each $x \in M$, the function of t, $\rho(\alpha_t(x))$, is continuous on $\mathbb{R}$. In recent years, we have investigated the structure of the subspace of M, $H^\infty(M,\alpha)$, which is defined to be

$$\{ x \in M: \ \rho(\alpha_t(x)) \in H^\infty(\mathbb{R}), \ \text{for all} \ \rho \in M_*\},$$

where $H^\infty(\mathbb{R})$ is the classical Hardy space consisting of the boundary values of functions bounded analytic in the upper half-plane. As in [4, 8, etc.], the elements of $H^\infty(M,\alpha)$ are called analytic with respect to $\{\alpha_t\}_{t\in\mathbb{R}}$ and $H^\infty(M,\alpha)$ itself, is called the analytic subalgebra of M determined by $\{\alpha_t\}_{t\in\mathbb{R}}$. Further, as in [4], $H^\infty(M,\alpha)$ is equal to the set of elements of M such that $Sp_\alpha(x) \subset [0, \infty)$ where $Sp_\alpha(x)$ is the Arveson spectrum of x with respect to $\{\alpha_t\}_{t\in\mathbb{R}}$ (cf. [1], [4]).

In this paper, we contribute a partial answer to the following

Question. When is $H^{\infty}(M,\alpha)$ maximal among the σ-weakly closed subalgebras of M?

For recent years, we have proved the partial answers of this question (cf. [5, 6, 7, 8, 11, 12, 13, 14, etc.]). In particular, Muhly and the second author in [8] proved that, if M is a crossed product determined by a von Neumann algebra N and a σ-weakly continuous flow $\{\beta_t\}_{t\in\mathbb{R}}$ on N and if $\{\alpha_t\}_{t\in\mathbb{R}}$ is the dual action of $\{\beta_t\}_{t\in\mathbb{R}}$, then $H^{\infty}(M,\alpha)$ is maximal among the σ-weakly closed subalgebras of M if and only if the fixed point algebra M^{α} (= N) is a factor. Recall that, if $\{\alpha_t\}_{t\in\mathbb{R}}$ is a dual action, then $\{\alpha_t\}_{t\in\mathbb{R}}$ is integrable in the sense of Connes-Takesaki [2]. Therefore, our aim in this note is to prove the following

Theorem. If $\{\alpha_t\}_{t\in\mathbb{R}}$ is integrable on M, then the fixed point algebra M^{α} is a factor if and only if $H^{\infty}(M,\alpha)$ is maximal among the σ-weakly closed subalgebras of M.

After finishing this note, we found the paper by Solel in [15] to study the maxility of $H^{\infty}(M,\alpha)$ in the general setting. However, we believe that our theory is interesting from a point of view of studying the structure of integrable actions in von Neumann algebras.

2. Preliminaries.

Let M be a von Neumann algebra on a Hilbert space H and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a σ-weakly continuous flow on M. First, we define the notion of spectral subspaces defined by [1]. We consider

$$\alpha(f)x = \int_{\mathbb{R}} f(t)\alpha_t(x)dt; \quad x \in M, \quad f \in \mathbb{L}^1(\mathbb{R}).$$

For $\mathbb{L}^1(\mathbb{R})$, we denote by $Z(f)$ the set $\{t \in \mathbb{R}: \hat{f}(t) = 0\}$, where $\hat{f}(t) = \int_{\mathbb{R}} e^{-ist}f(s)ds$. For $x \in M$, we define $Sp_\alpha(x)$ to be the set

$$\cap \{Z(f): f \in \mathbb{L}^1(\mathbb{R}), \quad \alpha(f)x = 0\}$$

and, for any closed subset S of $\mathbb{R}$, we define the spectral subspace $M^\alpha(S)$ to be $\{x \in M: Sp_\alpha(x) \subset S\}$. If S is not closed, then $M^\alpha(S)$ is defined to be the σ-weak closure of the set $\{x \in M: Sp_\alpha(x) \subset S\}$. We refer the reader to [1], [4] and [16] for the basic facts about spectra.

In this note, we write $H^\infty(M,\alpha)$ for $M^\alpha(\mathbb{R}_+)$ and $H_0^\infty(M,\alpha)$ for $M^\alpha(\mathbb{R}_{+0})$, where $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{+0} = (0, \infty)$, respectively. Further, we write M_t for $M^\alpha(\{t\})$ and note that

$$M^\alpha(\{t\}) = \{x \in M: \alpha_s(x) = e^{its}x, \quad s \in \mathbb{R}\}.$$

In particular, put $M^\alpha = M^\alpha(\{0\})$.

Let $\mathfrak{N}$ be the set of all $x \in M$ such that there is some $y \in M$ with $y = \int_{\mathbb{R}} \alpha_t(x^*x)dt$. If the linear span of $\mathfrak{N}$ is σ-weakly dense in M, we shall say that $\{\alpha_t\}_{t\in\mathbb{R}}$ is integrable. As in [2, 16], note that $\{\alpha_t\}_{t\in\mathbb{R}}$ is integrable if and only if $\int_{\mathbb{R}} \alpha_t(x)dt$, $x \in$

M_+, is a faithful normal semifinite operator valued weight on M(cf.[16]). Then we have the following lemma by [16, 21.4 Corollary].

Lemma 1. If $\{\alpha_t\}_{t\in\mathbb{R}}$ is integrable on M, then M is the von Neumann algebra generated by $\{M_t\}_{t\in\mathbb{R}}$ and $H^\infty(M,\alpha)$ is a σ-weakly closed subalgebra of M generated by $\{M_t\}_{t\in\mathbb{R}_+}$.

Let M be a von Neumann algebra on a Hilbert space H and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a σ-weakly continuous flow on M. Put $\tilde{M} = M \otimes B(L^2(\mathbb{R}))$ and let $\tilde{\alpha}_t = \alpha_t \otimes \mathrm{id}$. Then we easily have the following proposition

Proposition 2. Keep the notations as above. Then,

(i) for every subset S of $\mathbb{R}$, $\tilde{M}^{\tilde{\alpha}}(S) = M^\alpha(S) \otimes B(\mathbb{L}^2(\mathbb{R}))$.

(ii) The mapping $A \to A \otimes B(\mathbb{L}^2(\mathbb{R}))$ defines bijective correspondence between the class of σ-weakly closed subspaces of M and the class of σ-weakly closed subspaces of $M \otimes B(\mathbb{L}^2(\mathbb{R}))$ with the form $A \otimes B(\mathbb{L}^2(\mathbb{R}))$, where A is a σ-weakly closed subspace of M.

(iii) $H^\infty(\tilde{M},\tilde{\alpha}) = H^\infty(M,\alpha) \otimes B(\mathbb{L}^2(\mathbb{R}))$.

(iv) $\tilde{M}^{\tilde{\alpha}} = M^\alpha \otimes B(\mathbb{L}^2(\mathbb{R}))$.

Proof. (i). Since $M \otimes B(\mathbb{L}^2(\mathbb{R}))$ consists of all operators $x = (x_{ij}) \in B(H \otimes \mathbb{L}^2(\mathbb{R}))$ with operators $x_{ij} \in M$, we may consider $\tilde{\alpha}_t(x) = (\alpha_t(x_{ij}))$. Thus, we have $\tilde{\alpha}(f)x = (\alpha(f)x_{ij})$. By the definition of spectra, if $\tilde{\alpha}(f)x = 0$, then $\alpha(f)x_{ij} = 0$ for all i, j. Thus, if $x \in \tilde{M}^{\tilde{\alpha}}(S)$, then $x_{ij} \in M^\alpha(S)$ for all i, j. Hence we have $\tilde{M}^{\tilde{\alpha}}(S) \subset M^\alpha(S) \otimes B(\mathbb{L}^2(\mathbb{R}))$. Since the converse inclusion is clear, we have (i).

(ii) is clear and, from (i), we have (iii) and (iv). This completes the proof.

By Proposition 2, we have the following corollary.

Corollary 3. Keep the notations as above. Then $H^\infty(M,\alpha)$ is maximal among the σ-weakly closed subalgebras of M if and only if $H^\infty(\tilde{M},\tilde{\alpha})$ is maximal among the σ-weakly closed subalgebras of $\tilde{M}$.

Next, we recall that the crossed product $M \rtimes_\alpha \mathbb{R}$ determined by M and $\{\alpha_t\}_{t\in\mathbb{R}}$ is the von Neumann algebra on the Hilbert space $\mathbb{L}^2(\mathbb{R}, H)$ generated by the operators $\pi^\alpha(x)$, $x \in M$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by the equations

$$(\pi^\alpha(x)f)(t) = \alpha_{-t}(x)f(t), \quad f \in \mathbb{L}^2(\mathbb{R}, H), \quad t \in \mathbb{R},$$

and

$$(\lambda(s)f)(t) = f(t-s), \quad f \in \mathbb{L}^2(\mathbb{R}, H), \quad t \in \mathbb{R}.$$

The automorphism group $\{\hat{\alpha}_t\}_{t\in\mathbb{R}}$ of $M \rtimes_\alpha \mathbb{R}$ which is dual to $\{\alpha_t\}_{t\in\mathbb{R}}$ is implemented by the unitary representation of $\mathbb{R}$, $\{S_t\}_{t\in\mathbb{R}}$, defined by the formula

$$(S_t f)(s) = e^{ist}f(s), \quad f \in L^2(\mathbb{R}, H).$$

Further, we recall that the double crossed product $(M \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}$ is the von Neumann algebra on $\mathbb{L}^2(\mathbb{R}, \mathbb{L}^2(\mathbb{R}, H))$ generated by the operators $\pi^{\hat{\alpha}}(y)$, $y \in M \rtimes_{\alpha} \mathbb{R}$, and $\mu(s)$, $s \in \mathbb{R}$, defined by the equations

$$(\pi^{\hat{\alpha}}(y)g)(t) = \hat{\alpha}_{-t}(y)g(t), \quad g \in \mathbb{L}^2(\mathbb{R}, \mathbb{L}^2(\mathbb{R}, H)), \quad t \in \mathbb{R},$$

and

$$(\mu(s)g)(t) = g(t-s), \quad g \in \mathbb{L}^2(\mathbb{R}, \mathbb{L}^2(\mathbb{R}, H)), \quad t \in \mathbb{R}.$$

The automorphism group $\{\hat{\hat{\alpha}}_t\}_{t\in\mathbb{R}}$ of $(M \rtimes_{\alpha} \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}$ which is dual to $\{\hat{\alpha}_t\}_{t\in\mathbb{R}}$ is implemented by the unitary representation of $\mathbb{R}$, $\{S_t\}_{t\in\mathbb{R}}$, lefined by the formula

$$(\tilde{S}_t g)(s) = e^{ist}g(s), \quad g \in L^2(\mathbb{R}, \mathbb{L}^2(\mathbb{R}, H)).$$

For simplicity, we put $N = (M \rtimes_{\alpha} \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}$. From the definition of spectra, we have easily

Lemma 4. Let p be a projection of $M \rtimes_{\alpha} \mathbb{R}$. Put $\pi^{\hat{\alpha}}(p) = P$ and $\beta^P = \hat{\hat{\alpha}}_{|N_P}$, where N_P is the reduced von Neumann algebra of N by P. Then, for every subset S of $\mathbb{R}$, $(N_P)^{\beta^P}(S) = P\, N^{\hat{\hat{\alpha}}}(S)\, P$.

3. Proof of Theorem.

Keep the notations and the assumptions as in §2. Suppose that

$\{\alpha_t\}_{t\in\mathbb{R}}$ is integrable on M. Considering $\{M \otimes B(\mathbb{L}^2(\mathbb{R})), \alpha \otimes \mathrm{id}\}$, by Proposition 2 and Corollary 3, we may suppose that M^α is properly infinite to prove this theorem. By [10, Theorem 4.1], there exists a projection p in $(M \otimes B(\mathbb{L}^2(\mathbb{R})))^{\alpha\otimes \mathrm{Ad}(\rho)}$ $(= M \rtimes_\alpha \mathbb{R})$ such that

$$\{M \otimes B(\mathbb{L}^2(\mathbb{R})), \alpha \otimes \mathrm{id}\} \cong \{M \otimes B(\mathbb{L}^2(\mathbb{R})), \alpha \otimes \mathrm{Ad}(\rho)\}_p$$

where $\{\rho_t\}_{t\in\mathbb{R}}$ is the left regular representation of $\mathbb{R}$ on $\mathbb{L}^2(\mathbb{R})$ and $\mathrm{Ad}(\rho)$ is implemented by $\{\rho_t\}_{t\in\mathbb{R}}$. Put $P = \pi^{\hat{\alpha}}(p)$. From the duality theorem of crossed product, we have

$$\{M \otimes B(\mathbb{L}^2(\mathbb{R})), \alpha \otimes \mathrm{id}\} \cong \{N_P, \hat{\hat{\alpha}}_{|N_P}\},$$

where N_P is the reduced von Neumann algebra of N by P. Put $\beta^P = \hat{\hat{\alpha}}_{|N_P}$. That is, there exists an isomorphism Φ of $M \otimes B(\mathbb{L}^2(\mathbb{R}))$ onto N_P such that

$$\Phi \circ \tilde{\alpha}_t = \beta^P_t \circ \Phi, \quad t \in \mathbb{R}.$$

Then, for any $X \in N_P$ and $f \in \mathbb{L}^1(\mathbb{R})$, we have

$$\Phi(\tilde{\alpha}(f)X) = \Phi(\int_{\mathbb{R}} f(t)\tilde{\alpha}_t(X)dt) = \int_{\mathbb{R}} f(t)\Phi(\tilde{\alpha}_t(X))dt$$

$$= \int_{\mathbb{R}} f(t)\beta^P_t(\Phi(X))dt = \beta^P(f)(\Phi(X)).$$

Thus, we have the following

Proposition 5. For every subset S of $\mathbb{R}$, $\Phi(\tilde{M}^{\tilde{\alpha}}(S)) = (N_p)^{\beta^p}(S)$.

Let $(M \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}_+$ be the σ-weakly closed subalgebra generated by $\pi^{\hat{\alpha}}(M \rtimes_\alpha \mathbb{R})$ and $\{\mu(t)\}_{t\in\mathbb{R}_+}$. As in [8], we call it the analytic crossed products determined by $M \rtimes_\alpha \mathbb{R}$ and $\{\hat{\hat{\alpha}}_t\}_{t\in\mathbb{R}}$. By [8, Proposition 5.1], three spaces $H^\infty(N,\hat{\hat{\alpha}})$, $H_0^\infty(N,\hat{\hat{\alpha}})$ and $(M \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}_+$ coincide. Then, by Lemma 4 and Propositions 2 and 5, we have

Proposition 6. (i) $\Phi(H^\infty(\tilde{M},\tilde{\alpha})) = H^\infty(N_p,\beta^p) = P\, H^\infty(N,\hat{\hat{\alpha}})\, P$.

(ii) $\Phi(\tilde{M}^{\tilde{\alpha}}) = (N_p)^{\beta^p} = P\pi^{\hat{\alpha}}(M \rtimes_\alpha \mathbb{R})P = \pi^{\hat{\alpha}}((M \rtimes_\alpha \mathbb{R})_p)$, where $(M \rtimes_\alpha \mathbb{R})_p$ is the reduced von Neumann algebra of $M \rtimes_\alpha \mathbb{R}$.

To prove Theorem, by Proposition 6, it is sufficient to prove that $(M \rtimes_\alpha \mathbb{R})_p$ is a factor if and only if $H^\infty(N_p,\beta^p)$ $(= PH^\infty(N,\hat{\hat{\alpha}})P)$ is maximal among the σ-weakly closed subalgebras of N_p. Let $c(p)$ be the central projection of p in $M \rtimes_\alpha \mathbb{R}$. Then we have $(M \rtimes_\alpha \mathbb{R})_p{}'$ $= ((M \rtimes_\alpha \mathbb{R})')_p$ and $(M \rtimes_\alpha \mathbb{R})_{c(p)}{}' = ((M \rtimes_\alpha \mathbb{R})')_{c(p)}$. Since $((M \rtimes_\alpha \mathbb{R})')_p$ is isomorphic to $((M \rtimes_\alpha \mathbb{R})')_{c(p)}$, $(M \rtimes_\alpha \mathbb{R})_p$ is a factor if and only if $(M \rtimes_\alpha \mathbb{R})_{c(p)}$ is a factor.

Suppose that M^α is a factor, that is, $(M \rtimes_\alpha \mathbb{R})_{c(p)}$ is a factor. This implies that $c(p)$ is a minimal projection in the center $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})$ of $M \rtimes_\alpha \mathbb{R}$. Since $\hat{\alpha}_t(c(p))$ is a minimal projection in $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})$ for all $t \in \mathbb{R}$, $\hat{\alpha}_t(c(p))c(p) = 0$ or $c(p)$.

Since $\{\hat{\alpha}_t\}_{t\in\mathbb{R}}$ is σ-weakly continuous, $\hat{\alpha}_t(c(p))$ converges to p σ-weakly as $t \to 0$. It follows that $\hat{\alpha}_t(c(p)) = c(p)$ for all t in a neighborhood of 0 and, therefore, for all $t \in \mathbb{R}$. Put $Q = \pi^{\hat{\alpha}}(c(p))$. Then we have

$$\mu(t)Q\mu(t)^* = \mu(t)\pi^{\hat{\alpha}}(c(p))\mu(t)^* = \pi^{\hat{\alpha}}(\hat{\alpha}_t(c(p))) = \pi^{\hat{\alpha}}(c(p)) = Q.$$

This implies that Q is in the center $\mathfrak{Z}(N)$ of N. Since the reduced von Neumann algebra N_Q is generated by $\pi^{\hat{\alpha}}((M \rtimes_\alpha \mathbb{R})_{c(p)})$ and $\mu(t)Q$, we have $N_Q \cong (M \rtimes_\alpha \mathbb{R})_{c(p)} \rtimes_\gamma \mathbb{R}$, where $\gamma = \hat{\alpha}|(M \rtimes_\alpha \mathbb{R})_{c(p)}$ and the crossed product $(M \rtimes_\alpha \mathbb{R})_{c(p)} \rtimes_\gamma \mathbb{R}$ is considered on the Hilbert space $L^2(\mathbb{R}, c(p)\mathbb{L}^2(\mathbb{R}, H))$. Since $(M \rtimes_\alpha \mathbb{R})_{c(p)}$ is a factor, by [8, Theorem 5.2], $H^\infty(N,\hat{\alpha})Q$ (= $H^\infty(N_Q,\beta^{c(p)})$) is maximal among the σ-weakly closed subalgebras of N_Q.

We now prove that $H^\infty(N_P,\beta^P)$ is maximal among the σ-weakly closed subalgebras of N_P. Let B be a σ-weakly closed subalgebra of N_P containing $H^\infty(N_P,\beta^P)$ properly. We construct the σ-weakly closed subalgebra $\tilde{B}$ of N_Q generated by $H^\infty(N,\hat{\alpha})Q$ and B. Since $\tilde{B} \supsetneqq H^\infty(N,\hat{\alpha})Q$ clearly, we have $\tilde{B} = N_Q$. It is clear that $P\tilde{B}P = B$ and $(N_Q)_P = N_P$. Thus, $B = N_P$. Therefore, $H^\infty(N_P,\beta^P)$ is maximal among the σ-weakly closed subalgebras of N_P and so $H^\infty(M,\alpha)$ is maximal among the σ-weakly closed subalgebras of M.

Conversely, we suppose that $H^\infty(M,\alpha)$ is maximal among the σ-weakly closed subalgebras of M, that is, we suppose that $H^\infty(N_P,\beta^P)$ is maximal among the σ-weakly closed subalgebras of N_P.

Further, suppose that $(M \rtimes_\alpha \mathbb{R})_p$ is not a factor. Let $c(p)$ be the central projection of p in $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})$. Put $q = \bigvee_{t \in \mathbb{R}} \hat{\alpha}_t(c(p))$. Then $\hat{\alpha}_t(q) = q$ and so $\pi^{\hat{\alpha}}(q) \in \mathfrak{Z}(N)$. Putting $Q = \pi^{\hat{\alpha}}(q)$, then N_Q is isomorphic to the crossed product $(M \rtimes_\alpha \mathbb{R})_q \rtimes_\gamma \mathbb{R}$ defined by $(M \rtimes_\alpha \mathbb{R})_q$ and γ_t $(= \hat{\alpha}|_{(M \rtimes_\alpha \mathbb{R})_q})$ in such a way that $H^\infty(N_Q, \beta^q)$ is carried onto the analytic crossed product $(M \rtimes_\alpha \mathbb{R})_q \rtimes_\gamma \mathbb{R}_+$.

If $\{\gamma_t\}_{t \in \mathbb{R}}$ is not ergodic on $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_q$, then there exists a $\{\gamma_t\}_{t \in \mathbb{R}}$-invariant projection p_1 in $(M \rtimes_\alpha \mathbb{R})_q$ such that $0 \lneqq p_1 \lneqq q$. Since q is the least, $\{\gamma_t\}$-invariant central projection in $(M \rtimes_\alpha \mathbb{R})_q$ containing p, it is clear that $0 \lneqq p_1 p \lneqq p$. Put

$$\tilde{B} = \pi^{\hat{\alpha}}(p_1) H^\infty(N_Q, \beta^q) \oplus \pi^{\hat{\alpha}}(q - p_1) N_Q.$$

Then $\tilde{B}$ is a proper σ-weakly closed subalgebra of N_Q containing $H^\infty(N_Q, \beta^q)$ properly. Put $B = \pi^{\hat{\alpha}}(p) \tilde{B} \pi^{\hat{\alpha}}(p)$. If $B = H^\infty(N_P, \beta^P)$, then we have

$$\pi^{\hat{\alpha}}(p) \pi^{\hat{\alpha}}(q - p_1) \pi^{\hat{\alpha}}(M \rtimes_\alpha \mathbb{R}) \mu(t) \pi^{\hat{\alpha}}(p) = 0, \quad \text{for all} \quad t < 0.$$

Thus, $\pi^{\hat{\alpha}}(p) \pi^{\hat{\alpha}}(q - p_1) \mu(t) \pi^{\hat{\alpha}}(p) = 0$ for all $t < 0$ and so $(p - pp_1) \hat{\alpha}_t(p) = 0$ for all $t < 0$. Since $\{\hat{\alpha}_t\}_{t \in \mathbb{R}}$ is σ-weakly continuous, we have $(p - pp_1)p = 0$ and so $p = pp_1$. This is a contradiction. Then $B \neq H^\infty(N_P, \beta^P)$. Similarly, we have $B \neq N_P$. Therefore B is a properly σ-weakly closed subalgebra of N_P containing $H^\infty(N_P, \beta^P)$ properly. This is a contradiction. Consequently, without loss of generality, we may suppose that

$\{\gamma_t\}_{t\in\mathbb{R}}$ is ergodic on $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_q$. Then we need the following lemma as in [8].

Lemma 7. If $(M \rtimes_\alpha \mathbb{R})_p$ is not a factor and if $\{\gamma_t\}_{t\in\mathbb{R}}$ acts ergodically on the center $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_q$ of $(M \rtimes_\alpha \mathbb{R})_q$, then there is a strongly continuous family $\{e_t\}_{t<0}$ of projections in $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_q$ such that

$$e_{t+s} = e_t\gamma_t(e_s), \quad s, t < 0,$$

and $0 \lneqq e_t p \leqq e_0 p \lneqq p$ for some $t < 0$, where $e_0 = \operatorname{s-lim}_{t\uparrow 0} e_t$.

Proof. As in [8, Lemma 5.6], we note that $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_q$ is nonatomic and that there exists a faithful normal state on $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})$. By Cohen's factorization theorem,

$$\{\gamma(f)x\colon f \in \mathbb{L}^1(\mathbb{R}), \quad x \in \mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_q\}$$

is a $\{\gamma_t\}_{t\in\mathbb{R}}$-invariant, σ-weakly dense, C^*-subalgebra of $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_q$ on which $\{\gamma_t\}_{t\in\mathbb{R}}$ is strongly continuous. If Ω is the maximal ideal space of this subalgebra, then there is a continuous one-parameter group of homeomorphisms, $\{T_t\}_{t\in\mathbb{R}}$, of Ω, and, there is a nonatomic, quasi-invariant, ergodic, probability measure μ on Ω, with $\operatorname{supp}(\mu) = \Omega$, such that

$$\Gamma(\gamma_t(x))(\omega) = \Gamma(x)(T_t\omega) \quad \text{a.e.}(\mu),$$

where Γ is the canonical extension of the Gelfand transform to all of $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_q$, mapping isometrically onto $\mathbb{L}^\infty(\Omega, \mu)$. Since $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_p$ is isomorphic to $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_{c(p)}$, and, since $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_p$ is not a factor, there exists a projection e in $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_q$ such that $0 \lneqq ec(p) \lneqq c(p)$. Then there is a measurable subset E of Ω such that $\Gamma(e) = 1_E$. Since μ is regular on Ω, we may suppose that E is open in Ω. As in [8, Lemma 5.6], for each $t < 0$, put $E_t = \bigcap_{t \leqq s \leqq 0} T_s E$. If we define $e_t = \Gamma^{-1}(1_{E_t})$, $t < 0$ and $e_0 = \operatorname{s-lim}_{t \uparrow 0} e_t$, then $e_0 \leqq e$. Then we obtain the desired property of $\{e_t\}_{t<0}$. This completes the proof.

If $\{\gamma_t\}_{t\in\mathbb{R}}$ is ergodic on $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_q$, then, by Lemma 7, there exists a strongly continuous family, $\{e_t\}_{t<0}$, of projections in $\mathfrak{Z}(M \rtimes_\alpha \mathbb{R})_q$ such that

$$e_{t+s} = e_t\gamma_t(e_s), \text{ for all } s, t < 0,$$

and $0 \lneqq e_t p \leqq e_0 p \lneqq p$ for some $t < 0$, where $e_0 = \operatorname{s-lim}_{t\uparrow 0} e_t$. Let $\tilde{B}$ denote the σ-weak closure of the linear span of $H^\infty(N_Q, \beta^q)$ and $\{\pi^{\hat{\alpha}}(e_t)\pi^{\hat{\alpha}}(M \rtimes_\alpha \mathbb{R})\mu(t)\}_{t<0}$. Then, as in the proof of [8, Theorem 5.2], $\tilde{B}$ is a properly, σ-weakly closed subalgebra of N_Q containing $H^\infty(N_Q, \beta^q)$ properly. Put $B = \pi^{\hat{\alpha}}(p)\tilde{B}\pi^{\hat{\alpha}}(p)$. If $B = H^\infty(N_P, \beta^p)$, then

$$\pi^{\hat{\alpha}}(p)\pi^{\hat{\alpha}}(e_t)\pi^{\hat{\alpha}}(M \rtimes_\alpha \mathbb{R})\mu(t)\pi^{\hat{\alpha}}(p) = \{0\} \quad \text{for all } t < 0.$$

and so $\pi^{\hat{\alpha}}(p)\pi^{\hat{\alpha}}(e_t)\mu(t)\pi^{\hat{\alpha}}(p) = 0$ for all $t < 0$. Thus we have

$pe_t\hat{\alpha}_t(p) = 0$ for all $t < 0$. As $t \uparrow 0$, $pe_0p = e_0p = 0$. This is a contradiction. This implies that $B \supsetneqq H^{\infty}(N_p,\beta^p)$. On the other hand, if $B = N_p$, then we have for all $t < 0$,

$$\pi^{\hat{\alpha}}(p)\pi^{\hat{\alpha}}(e_t)\pi^{\hat{\alpha}}(M \rtimes_\alpha \mathbb{R})\mu(t)\pi^{\hat{\alpha}}(p) = \pi^{\hat{\alpha}}(p)\pi^{\hat{\alpha}}(M \rtimes_\alpha \mathbb{R})\mu(t)\pi^{\hat{\alpha}}(p),$$

and so, multipling both left side by $\pi^{\hat{\alpha}}(q-e_t)$, we have

$$\pi^{\hat{\alpha}}(q-e_t)\pi^{\hat{\alpha}}(p)\pi^{\hat{\alpha}}(M \rtimes_\alpha \mathbb{R})\mu(t)\pi^{\hat{\alpha}}(p) = 0 \quad \text{for all} \quad t < 0.$$

Therefore, we have $(q-e_t)p\hat{\alpha}_t(p) = 0$ for all $t < 0$. As $t \uparrow 0$, $p-pe_0 = 0$. This contradiction implies that $B \subsetneqq N_p$. This implies that $H^{\infty}(N_p,\beta^p)$ is not maximal among the σ-weakly closed subalgebras of N_p. This is a contradiction. Therefore, $(M \rtimes_\alpha \mathbb{R})_p$ is a factor and so M^α is a factor. This completes the proof of Theorem.

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