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The Limit of the Application of Runge-Kutta Method
(Runge-Kutta 法の限界)

Hideo Yamagata
(阪府大工　山縣 秀雄)

Introduction. When we solve an initial value problem of a second order ordinary differential equation numerically, we rewrite the equation to equivalent two first order simultaneous differential equations and usually use famous Runge-Kutta-Nyström method. But we cannot find the limit of the application of the method in any articles and books.

Which equation can be solved by the method? It is a question. Here we would like to introduce two equations in Examples 1 and 2. These equations satisfy Lipschitz condition. But when we want to solve the initial value problems of the equations numerically, it seems to be difficult to apply the Runge-Kutta method.

1. Two examples.

Example 1. Let us consider the second order ordinary nonlinear differential equation having the solution

$$y(x)=(1/x)\sin\{(1/x)+\delta\}, \qquad (1)$$

where δ is a constant. Since the relation

$$x^2y'+xy=(1/x)\cos\{(1/x)+\delta\}, \qquad (2)$$

holds for the solution (1), $1/x^2$ is expressed by

$$1/x^2=y^2+(x^2y'+xy)^2, \qquad (3)$$

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Department of Math. Sci. College of Engineering, University of Osaka Prefecture, Mozu, Sakai City, Osaka, Japan.

and second derivative

$$y''=(2/x^2-1/x^4)y+(4/x^2)(xy'+y) \qquad (4)$$

leads to the equation

$$y''=\{y^2+(x^2y'+xy)^2\}\{4xy'+6y-y^3-x^2y(xy'+y)^2\}. \qquad (5)$$

The second order nonlinear equation (5) can be also expressed by the equivalent two simultaneous equations

$$\begin{cases} y'=z, \\ z'=\{y^2+(x^2z+xy)^2\}\{4xz+6y-y^3-xy(xz+y)^2\}, \end{cases} \qquad (6)$$

satisfying Lipschitz condition, and Runge-Kutta method can be formally applied to the system (6).

Let x_0 be a negative fixed value. When we take

$$\begin{cases} y(x_0)=(1/x_0)\sin\{(1/x_0)+\delta\}, \\ y'(x_0)=-(1/x_0^2)\sin\{(1/x_0)+\delta\}-(1/x_0^3)\cos\{(1/x_0)+\delta\}, \end{cases} \qquad (7)$$

as the initial condition of equation (5), the solution of the equation is divergent at x=0. Because x=0 is an essentially singular point of the solution (1). We are interested in the numerical solution of the equation (5) with the initial value (7) solved by the Runge-Kutta-Nyström method.

Next we would like to show another example of the second order ordinary nonlinear autonomous differential equation having the same property. The equation also satisfies the Lipschitz condition, and all nontrivial solutions of the autonomous equation has an essential singularity similar to the one in the solution (1).

Example 2. Let us consider the second order autonomous equation having the solution

$$y=(1/x)\sin(\log_e K|x|), \tag{8}$$

where K is a positive constant. Since the relation

$$xy'+y=(1/x)\cos(\log_e K|x|) \tag{9}$$

holds for the solution (8), $1/x^2$ is expressed by

$$1/x^2=y^2+(xy'+y)^2\ . \tag{10}$$

We also have the second derivative

$$y''=(-3/x)y'-(2/x^2)y. \tag{11}$$

Setting $1/x=t$ in the equation (10), we have an algebraic equation

$$t^4-2y^2t^2-2yy't-y'^2=0 \tag{10'}$$

with order 4. It is solvable by radicals. We find a root $t=f(y,y')\leqslant 0$ of the equation (10')taking the nonpositive real values. Substituting the root $f(y,y')$ into the second derivative (11), we have the equation

$$y''=-3f(y,y')\cdot y'-2y\cdot f(y,y')^2\ . \tag{12}$$

Since two functions $f(y,y')\cdot y'$ and $f(y,y')^2$ satisfy the Lipschitz condition, the equation (12) also satisfy the Lipschitz condition.

Remark 1. In Example 1, the algebraic equation corresponding (10') is

$$t^6-y^2t^4-(y'+yt)^2=0, \tag{3'}$$

with order 6.

2. Lipschitz Condition.

Here, we would like to show the various Properties of the function $f(y,y')$. Let us denote the function

$$t^4-2y^2t^2-2yzt-z^2$$

by $F(t,y,z)$. That is,

$$F(t,y,z)\equiv t^4-2y^2t^2-2yzt-z^2. \tag{13}$$

The left hand side of the equation (10') is $F(t,y,y')$.

Proposition 1. The real root $t=f(y,z)$ of the equation

$$F(t,y,z)=0, \tag{14}$$

is a C^∞-function of (y,z), provided that the condition $(z, yt)\neq(0,0)$ holds.

Proof. If two equations

$$F(t,y,z)=0, \quad \text{and } F_t(t,y,z)=4t^3-4y^2t-2yz=0$$

hold for a (t,y,z) simultaneously, the equation

$$t\,F_t(t,y,z)-4F(t,y,z)=4y^2t^2+6yzt+4z^2=0 \tag{15}$$

also holds. The equation

$$3(yt+z)^2+(y^2t^2+z^2)=0 \tag{16}$$

is derived from the equation (15), and it follows from the Implicite function theorem and the equation (16) that the real root $t=f(y,z)$ is a C^∞-function except for the case $z=yt=0$.

Proposition 2. If the condition $yz\neq0$ holds, any root $t=f(y,z)$ of the equation (14) is a simple root of the equation.

Proof. If the equation (14) has a double root, it satisfies the equqtion (15), and is equal to one of

$$t_\pm=\{(-3\pm\sqrt{7}i)/4\}\cdot(z/y). \tag{17}$$

Since

$$F_t(t_\pm, y, z) = -(z/y)\{(-3 \pm \sqrt{7}i)(4y^4+13z^2)/4+(2y^4+6z^2)\},$$

and since the roots (17) do not satisfy the equation

$$F_t(t,y,z)=0,$$

then any root $t=f(y,z)$ of the equation (14) is a simple root under the condition $yz \neq 0$.

We easily obtain the following four results:

(i) When $z=0$ holds, we have four roots $t=\pm\sqrt{2}y$ and $t=0$ (double root) of the equation (14).

(ii) When $y=0$ holds, we have four roots $t=\pm\sqrt{|z|}$ and $t=\pm\sqrt{|z|}\, i$ of the equation (14).

(iii) $t=0$ leads to $z=0$. Then $z \neq 0$ leads to $t \neq 0$.

(iv) The root t depends continuously to (y,z).

Then if $z \neq 0$ holds, the equation (14) has a positive root, a negative root and a pair of conjugate complex roots. The conjugate complex roots tend to zero as z tends to zero.

Next we would like to examine the behaviour of the root $t=f(y,z)$ of the equation (14) in a neighbourhood of $(y,z)=(0,0)$.

Proposition 3. Let $t=f(y,z)$ be a real root of the equation (14). Then two functions $\{f(y,z)\}^2$ and $zf(y,z)$ satisfy the local Lipschitz condition at $(y,z)=(0,0)$.

Proof. Equation (14) is expressed by the equation

$$(y,z)\begin{pmatrix} 2t^2 & t \\ t & 1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = t^4. \qquad (18)$$

Since the proper equation

$$\begin{vmatrix} 2t^2-\lambda & t \\ t & 1-\lambda \end{vmatrix} = 0 \qquad (19)$$

has two positive roots $\lambda_{\pm} = (2t^2+1 \pm\sqrt{4t^4+1}\,)/2$,

the equation (18) with a fixed $t \neq 0$ gives a ellipse.

Since $(t^2/\sqrt{\lambda_+}\,)^2 = t^4 \cdot 2/(2t^2+1+\sqrt{4t^4+1}\,)$

$$= t^2(2t^2+1-\sqrt{4t^4+1}\,)/2$$

and $\lim_{t\to 0} t^2/(t^2/\sqrt{\lambda_+}) = \lim_{t\to 0} t^2/\sqrt{t^2(2t^2+1-\sqrt{4t^4+1})/2} = 1$

hold, the real valued function $f(y,z)$ satisfies local Lipschitz condition with Lipschitz constant $1+\varepsilon$ ($\varepsilon>0$) at $(y,z)=(0,0)$. Since the real root $f(y,z)$ is bounded in a neighbourhood of $(y,z)=(0,0)$, the function $zf(y,z)$ also satisfies local Lipschitz condition at $(y,z)=(0,0)$.

3. General solution.

Let $y(x;c,K)$ denote the solution

$$y=\{\sin(\log_e K|x-c|\,)\}/(x-c) \quad (\text{for } x<c) \qquad (20)$$

of the second order differential equation (12), x_0 a fixed real value and $S(x_0)$ the set

$\{(y(x_0;c,K),\ y'(x_0;c,K));\ c>x_0,\ K>0\}\ (\subseteq R^2)$.

Since the relation

$$\bigcup_{t\geq 0}\left\{(y,z);\quad (y,z)\begin{pmatrix} 2t^2 & t \\ t & 1\end{pmatrix}\begin{pmatrix} y \\ z\end{pmatrix} = t^4\right\} = R^2,$$

given from (18), holds, the relation

$$S(x_0)\cup\{(0,0)\} = R^2$$

also holds for any fixed real value x_0.

Then the solutions of the second order differential equation (12) consist of the general solution

$$y=y(x;c,K) \tag{20}$$

and a singular solution $y\equiv 0$, where $K>0$ and c,K are constants. Thus it is difficult to solve the equation (12) by Euler method or Runge-Kutta method. To solve the equation (14) it is required to obtain square root and cube root of positive numbers.

4. Other singular solution of the second order equation. P. Painleve[1] treated the equation

$$y''=F(x,y,y') \tag{21}$$

with the rational function $F(x,y,y')$ of x,y and y'. The nonlinear second order ordinary differential equation satisfying Lipschitz condition has also various singular solutions of other types. For example the equation

$$y''=2y'^3 \tag{22}$$

has the solution $y=\sqrt{-x+c_1}+c_2$, and the equation

$$y''=y'^2 \tag{23}$$

has the solution $y=\log_e|t+c|$ [2] p.816, where c_1, c_2 and c are constants.

Conclusion. When we solve an initial value problem of the second order ordinary differential equation numerically, we must confirm the various conditions in [2], [3], [4] and [5] besides the Lipschitz condition. So far as we solve an initial value problem of the equation (12), we must use Euler method and estimate the behaviour of the solution roughly at the first step. It is a conclusion of the discussions with

Prof. S. Hitotsumatsu at the sumposium held at Kyoto University on 30th September 1988. When we calculate the values of the right hand side of the equation (12), we have accumulations of round-off errors due to obtaining square roots, cube roots, etc. Then we also have the following discussions there: whether the equation (12) must be regarded as an algebraic differential equation or not? I think that is depends on the size of the accumulations of errors.

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