ASYMMETRIC NEURAL NETWORKS
WITH EFFECTOR AND RECEPTOR PARAMETERS

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1. Introduction

Neural Networks now seem to be undergoing a renaissance since the works of Hopfield, his associates, and others (Hopfield, 1982, 1984; Hopfield and Tank, 1985; Rumelhart, McClelland and the PDP research group, 1986). Although the modeling nervous systems is as old as the era of McCulloch and Pitts (McCulloch and Pitts, 1943) and not much new things seem to be added to the old framework as far as the system definitions are concerned, what makes the new trend interesting is as follows. Hopfield was the first to show that there is a Lyapunov function for a given neural network which operates *asynchronously* where each component model neuron is basically that of McCulloch and Pitts and the connecting weights are *symmetric*. This made us possible at least partially to associate the equilibrium states of the neural network with our desired target states depending on the actual encode system of our application. One such example is the cerebrated travelling salesman problem (Hopfield and Tank, 1985) albeit some controversy has arisen recently (Wilson and Pawley, 1988).

The characteristic features of the Hopfield networks, i.e., symmetric weights and asynchronous operation, are at the same time somewhat unsatisfactory. The alleged proposition that the neural networks should have symmetric weights in order to have Lyapunov functions is rather restrictive especially when we have biological applications in mind. It is true that we can have a Lyapunov function for an asymmetric network regarding the average of two weights connecting a pair of elements as an equivalent symmetric weight, this means that we are changing the behavior of the component McCulloch-Pitts neurons. (See, for example, Feldman,

1987; Uesaka, 1987) As to the asynchronous mode of operation, the advantage of having distributed processors is largely wasted when we are concerned with the total operation time for arriving at stable states, if any. Then we need to find some way to escape from the symmetry restriction and also to pay more attention to the synchronous operation of neural networks. (See, for example, Little, 1974)

For that purpose, we analyse, in Section 2, some basic properties of the energy functions which are concomitant with the behavior of each component McCulloch-Pitts neuron. The main result there is that we can have Lyapunov functions for a slightly extended class of asymmetric neural networks (which we call *quasi-symmetric*).

In Section 3, we then introduce a special class of asymmetric neural networks in which each McCulloch-Pitts type neuron i has an effector parameter a_i and a receptor parameter b_i (which we call *e-r neural networks*). The effector parameter signifies the signal strength when the neuron fires and the receptor parameter denotes the sensitivity of the neuron when it receives a signal from other neurons. Thus the weight connecting from neuron j to neuron i is represented as $w_{ij} = b_i a_j c_{ij}$ which is asymmetric in general where $c_{ij} = 1$ if there is a connection from neuron j to i and $c_{ij} = 0$, otherwise.

Some analyses on the properties of such asymmetric networks follow under separate subsections. First, we analyse the global state transitions when the receptor parameters are positive. (Section 3.1.) Then the class of neural networks with effector and receptor parameters is shown to be a rich class in the sense that an arbitrary finite state transition can be embedded in the state transition of a certain neural network in the class. (Section 3.2.) In Section 3.3, we consider the case where the receptor parameters take any real values to obtain similar results as before.

Finally, we move to observe that a certain class of e-r neural networks can be viewed as quasi-symmetric networks. By extending Goles' result (Goles, 1987) for symmetric case, we

show in Section 4 that Lyapunov functions exist for the above mentioned class of asymmetric neural networks under synchronous as well as asynchronous modes of operation.

2. Network Definitions and Searching for Energy Functions

Consider a system of n McCulloch-Pitts neurons i (i = 1, 2, ..., n) where the state of the neuron i at time t, denoted as $s_i(t)$, takes the value 1 (firing) or 0 (resting). If we write the weight connecting i to j as w_{ji}, the next state of the neuron i can be defined as follows.

$$s_i(t+1) = H(\sum_{j=1}^{n} w_{ij} s_j(t) - \theta_i) \tag{2.1-a}$$

where θ_i is the threshold value for i, and H(x) is the Heaviside function defined as

$$H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{2.1-b}$$

Note that H(0) is undefined and we assume that $\sum_{j=1}^{n} w_{ij} s_j - \theta_i \neq 0$ for any $s_j \in \{0,1\}$ j = 1,2,...,n. Then we call this system a *neural network* (NN, for short) and denote as N(n, W, θ) where W = $[w_{ij}]_n$ and θ = $(\theta_1, \theta_2, ..., \theta_n)$. To define actual behaviors of an NN, we have to specify the mode of operation. If we pick a neuron and apply the above defined transition rule for it with all the other neurons in the same states, the NN is said to be operating *asynchronously*. On the other hand, if all the neurons in the system are applied the above rule simultaneously, it is operating *synchronously*. In both cases, the global transition functions can be defined as mappings from S to S where S is the set of state configurations s = $(s_1, s_2, ..., s_n)$, i.e., S = $\{0,1\}^n$. The global transition function for an NN N(n, W, θ) under synchronous operation is denoted as F_N. A state configuration s is said to be *cyclic* if $F_N^i(s)$ = s for some positive integer i; the minimum such i is called the cycle length. A *stable* state is the one which belongs to a cycle of length one.

An NN $N(n, W, \theta)$ is called *symmetric* if the weight matrix W is symmetric, i.e., if $w_{ij} = w_{ji}$ for $i, j = 1, 2, ..., n$. As is now well known, global state transitions of symmetric NNs have simple cycle structures : Under synchronous operation, they can have cycles each of whose length is at most two. (Goles, 1987) If systems operate asynchronously with additional condition that $w_{ii} = 0$ for $i = 1, 2, ..., n$, then every state transition path ends up with a stable state. (Hopfield, 1982) These results have been shown by devising the following Lyapunov functions respectively.

$$E_s(t) = - \sum_{i,j=1}^{n} w_{ij} s_i(t) s_j(t-1) + \sum_{i=1}^{n} (s_i(t) + s_i(t-1)) \theta_i \quad \text{(synchronous case)} \quad (2.2)$$

$$E_a(t) = - \sum_{i,j=1}^{n} w_{ij} s_i(t) s_j(t) + 2 \sum_{i=1}^{n} s_i(t) \theta_i \quad \text{(asynchronous case)} \quad (2.3)$$

First, we examine whether symmetric weights are necessary to have such energy functions as shown above. For that purpose, let us not give an energy function a priori at the outset but begin with considering the behavior of each constituent neuron . We assume, however, that there is some energy function E (which is a mapping from the set of state configurations into the set of real numbers) such that any proper state change at a neuron decreases its value. (Thus, we have asynchronous operation in mind here.)

Consider an NN $N(n, W, \theta)$ and a state configuration $s = (s_1, s_2, ..., s_n)$ where $s_i \in \{0,1\}$ for $i = 1, 2, ..., n$. If we focus our attention on the k-th neuron, the state transition is determined by evaluating $\sum_{j=1}^{n} w_{kj} s_j(t) - \theta_k = d_k$: $s_k(t+1)$ becomes 1 if d_k is positive and 0, if it is negative (by definition). This means that s_k should change to $\overline{s_k}$ if $(s_k - \overline{s_k}) d_k < 0$ where $\overline{s_k} = 1 - s_k$. Let s_k denote a state configuration which is the same as s except at neuron k, i.e., $s_k = (s_1, s_2, ..., \overline{s_k}..., s_n)$. Then the above observation motivates us to define

$$\Delta E_k(s) = E(s) - E(s_k) = - \widetilde{s_k} f_k(d_k) \quad (2.4)$$

where $\widetilde{s_k} = s_k - \overline{s_k}$ and f_k is a sign preserving monotone function. This $\Delta E_k(s)$ may be considered as a driving force to change the state of k-th neuron from s_k to $\overline{s_k}$.

We first note that $\Delta E_k(s) = -\Delta E_k(s_k)$ by definition, which yields the following.

$$\widetilde{s_k}f_k(d_k) = -\widetilde{\widetilde{s_k}}f_k(d_k + w_{kk}\widetilde{s_k})$$

If f_k is strictly increasing, then we have to have $w_{kk}\widetilde{\widetilde{s_k}} = 0$, which means $w_{kk} = 0$.

For $k \neq l$, we similarly define $\Delta E_{k,l}(\ s\ )$ as the difference between $E(\ s\ )$ and $E(\ s_{k,l}\ )$ where $s_{k,l} = (s_1, s_2,...,\overline{s_k},..,\overline{s_l},..s_n)$. Our convention is that $s_{k,l}$ is obtained from s by changing the states of k-th and l-th neurons in this order. Then we have

$$\Delta E_{k,l}(\ s\ ) \ = E(\ s\ ) - E(\ s_{k,l})$$
$$= E(\ s\ ) - E(\ s_k) + E(\ s_k\ ) - E(\ s_{k,l}\ )$$
$$= \Delta E_k(\ s\ ) + \Delta E_l(\ s_k\ ) \qquad (2.5)$$

By simple calculations, we get

$$\Delta E_l(\ s_k\ ) = -\ \widetilde{s_l}f_l(d_l - w_{lk}\widetilde{s_k}).$$

Thus,

$$\Delta E_{k,l}(\ s\ ) = -\ \widetilde{s_k}f_k(d_k) - \widetilde{s_l}f_l(d_l - w_{lk}\widetilde{s_k}) \qquad (2.6)$$

If we change the order of calculation, we have

$$\Delta E_{l,k}(\ s\ ) = -\ \widetilde{s_l}f_l(d_l) - \widetilde{s_k}f_k(d_k - w_{kl}\widetilde{s_l}). \qquad (2.7)$$

Since $s_{k,l} = s_{l,k}$ by definition, we should have $\Delta E_{k,l}(\ s\ ) = \Delta E_{l,k}(\ s\ )$, which means
$$\widetilde{s_k}(f_k(d_k) - f_k(d_k - w_{kl}\widetilde{s_l})) = \widetilde{s_l}(f_l(d_l) - f_l(d_l - w_{lk}\widetilde{s_k})).$$

In order to continue the calculation, we resort to a simplifying assumption that each f_k is a linear function, i.e., $f_k(x) = c_k x$ for a positive constant c_k. Then we have $c_k w_{kl} = c_l w_{lk}$. Thus, under the above mentioned assumption, the relations $c_k w_{kl} = c_l w_{lk}$ (k, l = 1,2,...,n) must hold for some positive constants c_k's (k = 1,2,...,n) in order for the network to have some noncontradictory energy function. We call these relations as *quasi-symmetric* because of the following lemma.

Lemma 2.1.

Let c_k (k = 1,2,..., n) and w_{ij} (i, j = 1,2,..., n) be real numbers. Then the following conditions are equivalent.

(1) $c_i w_{ij} = c_j w_{ji}$ for i, j = 1,2,..., n.

(2) $w_{ij} = v_{ij} c_j$ where v_{ij}'s are real numbers such that $v_{ij} = v_{ji}$ for i,j = 1,2,..., n .

Assuming the above mentioned quasi-symmetric condition and zero-diagonal condition, we can proceed to calculate, for example,

$$\Delta E_{k,l,m}(\mathbf{s}) = E(\mathbf{s}) - E(\mathbf{s}_{k,l,m})$$

$$= \Delta E_k(\mathbf{s}) + \Delta E_l(\mathbf{s}) + \Delta E_m(\mathbf{s}) + c_k\widetilde{w_{kl}} + c_l\widetilde{w_{lm}} + c_m\widetilde{w_{mk}} \qquad (2.8)$$

where $\widetilde{w_{kl}} = \tilde{s}_k\tilde{s}_l w_{kl}$ for $k,l = 1,2,...,n$.

In general, let I be a subset of $\{1,2,...,n\}$, then we have

$$\Delta E_I(\mathbf{s}) = \sum_{i\in I} \Delta E_i(\mathbf{s}) + \frac{1}{2}\sum_{i,j\in I} c_i\widetilde{w_{ij}} \qquad (2.9)$$

For a given $\mathbf{s} = (s_1,s_2,...,s_n)$, let $I(\mathbf{s})$ be the set of index i such that $s_i = 1$: $I(\mathbf{s}) = \{ i \mid s_i = 1\}$. Then $\Delta E_{I(\mathbf{s})}(\mathbf{s}) = E(\mathbf{s}) - E(\mathbf{s}_{I(\mathbf{s})})$.

By definition, $\mathbf{s}_{I(\mathbf{s})} = \mathbf{0} = (0,0,...,0)$ and let $E(\mathbf{0})$ be the reference value in evaluating $E(\mathbf{s})$. More simply put, assume that $E(\mathbf{0}) = 0$ and define $E(\mathbf{s}) = \Delta E_{I(\mathbf{s})}(\mathbf{s})$.

$$
\begin{aligned}
E(\mathbf{s}) &= \Delta E_{I(\mathbf{s})}(\mathbf{s}) \\
&= \sum_{i\in I(\mathbf{s})} \Delta E_i(\mathbf{s}) + \frac{1}{2}\sum_{i,j\in I(\mathbf{s})} c_i\widetilde{w_{ij}} \\
&= \sum_{i=1}^{n} \Delta E_i(\mathbf{s})\, s_i + \frac{1}{2}\sum_{i,j=1}^{n} c_i\widetilde{w_{ij}}s_i s_j \\
&= -\sum_{i=1}^{n} c_i d_i\, s_i + \frac{1}{2}\sum_{i,j=1}^{n} c_i w_{ij}s_i s_j \qquad (2.10)
\end{aligned}
$$

Substituting $d_i = \sum_{j=1}^{n} w_{ij}s_j - \theta_i$ we have

$$E(\mathbf{s}) = -\frac{1}{2}\sum_{i,j=1}^{n} c_i w_{ij}s_i s_j + \sum_{i=1}^{n} c_i s_i\theta_i \qquad (2.11)$$

Multiplying by two, for notational convenience, we have a desired Lyapunov function as below.

Theorem 2.2.

Let $N(n, W, \theta)$ be a quasi-symmetric and zero-diagonal NN where $w_{ij} = v_{ij}c_j$ such that $v_{ij} = v_{ji}$ and $c_j > 0$ for $i,j = 1,2,...,n$. Then under asynchronous operation mode, the following function is monotone non-increasing.

$$F_a(t) = -\sum_{i,j=1}^{n} c_i w_{ij}s_i(t)s_j(t) + 2\sum_{i=1}^{n} c_i s_i(t)\theta_i . \qquad (2.12)$$

3. Neural Networks with Effector and Receptor Parameters

Now we introduce the concept of effector and receptor parameters, by which asymmetric weights are defined in a very simple way. We assume that when a neuron i fires, the signal is assumed to be transmitted to each synapse with strength a_i which is called the effector parameter of the neuron. Our convention is that a_i is positive if the neuron is excitatory and is negative if it is inhibitory. Since we have only one effector parameter for each neuron, we are assuming that every synapse of a fixed neuron has the same signal transmission capability to every corresponding postsynaptic neuron. This is actually a gross simplification because the effect of a firing may vary depending on the shape and amount of each synaptic association. When a signal arrives, on the other hand, at a certain neuron through a synapse, we assume that the neuron has the sensitivity index in receiving the signal. That is, each neuron i has the receptor parameter b_i which denotes the effectiveness of transmitting a signal through each of its synaptic junctions. Again we assign a unique receptor value to each neuron, which disregards the actual differences of sensitivity among the types and locations of synaptic connections. Connections among neurons are specified by a connection matrix $C = [c_{ij}] : c_{ij} = 1$ if neuron j has synaptic connection with neuron i and $c_{ij} = 0$ if otherwise. We call a system composed of the above mentioned model neurons as a neural network with effector and receptor parameters as defined formally below.

Definition.

Let $N(n, W, \theta)$ be a neural network. It is called a *neural network with effector and receptor parameters* (e-r NN, for short) if there are real numbers a_i's (called effector parameters) and b_i's (called receptor parameters) for i = 1, 2, ..., n such that $w_{ij} = b_i a_j c_{ij}$ where c_{ij} takes the value 1 or 0. We denote such an e-r NN by $N(n, \mathbf{ba \bullet C}, \theta)$ where $\mathbf{a} = (a_1, a_2, ..., a_n)$, $\mathbf{b} = (b_1, b_2, ..., b_n)^t$ and $C = [c_{ij}]$.

In this section, we consider the global behavior of such e-r NN under *synchronous* operation mode. Then by substituting $w_{ij} = b_i a_j c_{ij}$ into equation (2.1-a) for every i, we have

$$s_i(t+1) = H(b_i\{\sum_{j=1}^{n} a_j c_{ij} s_j(t) - \frac{\theta_i}{b_i}\}) \qquad (i=1,2,...,n) \qquad (3.1)$$

This relation suggests that it would be convenient to rename the neurons so that they sort in order of θ_i/b_i. That is, we assume the following order relation throughout this paper and refer to it as the *normal ordering* of θ/b.

$$\frac{\theta_1}{b_1} \leq \frac{\theta_2}{b_2} \leq \cdots \cdots \leq \frac{\theta_n}{b_n} \qquad (3.2)$$

In the balance of this paper, we also assume that the connection matrix C is symmetric. In fact, we mostly treat the *full* case where $c_{ij} = 1$ for all i and j as in the following subsections. Thus in the full case, the connections are complete and we need not write C or c_{ij} explicitly.

3.1. Analysis of Global State Transitions for Simple Full e-r NN

Consider a full e-r NN $N(n, \mathbf{ba}, \theta)$, and assume that $b_i > 0$ for i = 1, 2, ..., n. This corresponds to the case where an excitatory neuron always yields positive connection weights and an inhibitory one, negative weights. Since it seems both quite natural and simple, we call such an e-r NN as *simple*. For a simple full e-r NN, we have

$$s_i(t+1) = H(\sum_{j=1}^{n} a_j s_j(t) - \frac{\theta_i}{b_i}) \qquad (i=1,2,...,n) \qquad (3.3)$$

By virtue of the normal ordering, we have the following property.

Lemma 3.1.

Let $N(n, \mathbf{ba}, \theta)$ be a simple e-r NN with normal ordering. Then we have the following relations.

If $s_i(t+1) = 0$, then $s_{i+1}(t+1) = 0$;

If $s_{i+1}(t+1) = 1$, then $s_i(t+1) = 1$ for $i = 1,2,..., (n-1)$.

Let $s(t) = (s_1(t),s_2(t),...,s_n(t))$ denote a state configuration of the system at time t. We sometimes omit the symbol (t) when there is no confusion and let s_k be a vector $(1,1,...,1,0,0,...0)$ where there are k consecutive 1's on the left side of the vector. We call these special configuration s_k's for a simple full e-r NN as *standard* for $k = 0,1,2,...,n$ where s_0 is the **0** vector $(0,0,...,0)$. Then what Lemma 3.1 says is that any configuration that is derivable from other state is standard.

Now, we are in a position to analyse the global state transition of a given simple e-r NN $N(n, \mathbf{ba}, \theta)$ defined in the previous section. Since the network has n neurons each having the state 0 or 1, there are 2^n states whose set is denoted as $S=\{(s_1,s_2,...,s_n) \mid s_i$ is 0 or 1 for $i=1,2,...n\}$. We divide S into the set of standard states S_{st} and the rest S_{ns}. Let [n] denote the set $\{0,1,2,...,n\}$. Then [n] and S_{st} are isomorphic by the obvious correspondence k in [n] with s_k in S_{st}. In this sense, we use k and s_k interchangeably when there is no confusion. Using the global state transition function $F_N: S \to S$ such that $F_N(s(t)) = s(t+1)$ as defined in (3.1), we can restate Lemma 3.1 as follows.

Theorem 3.2.

Let $N(n, \mathbf{ba}, \theta)$ be a simple e-r NN with normal ordering where the global state transition is denoted by F_N. Let S and S_{st} be the sets of all states and standard states, respectively. Then we have

$$S_{st} \supset F_N(S) .$$

In particular, $S_{st} \supset F_N(S_{st})$.

Thus any state in S_{ns} changes to a state in S_{st} in one step and the states in S_{st} are closed under the state transition. Then we first look into the transitions in S_{st}.

Consider a state k in S_{st}, and define $A_k = \sum_{j=1}^{k} a_j$. (Note that $A_0 = 0$ by definition.) A_k falls somewhere in the ordering relation of (3.2). That is, there is a unique integer u in [n] such that the following relation holds.

$$\frac{\theta_u}{b_u} < A_k < \frac{\theta_{u+1}}{b_{u+1}} \; .$$

In the above relation, disregard the left-hand side inequality when u = 0, and the right-hand side when u = n. This means that the next state is u, i.e., we have $F_N(k) = u$ in this case. In more general cases, we only have to consider various subsums of $\{a_1, a_2, ..., a_n\}$ instead of A_k's and find the corresponding place in the ordering relation (3.2).

By the above characterization of the global state transition of simple full e-r NN's, we are able to deduce the followings.

Lemma 3.3.

For the global state transitions of a simple full e-r NN with n neurons:

(1) Cyclic states and in particular, stable states, if any, are composed of the standard states. Thus the number of states which belong to one of the cycles is at most n+1.

(2) The maximum cycle length is at most n+1.

(3) The number of stable states is at most n+1.

(4) The length of the longest path (which is composed of one non-standard state and some of the standard states) to a cyclic state is at most n+1.

3.2. Synthesis of Simple Full e-r NN

Let $f: S \to S$ be a function where S is a finite alphabet. $N(n, W, \theta)$ with global transition function F_N is said to *realize* f if there exists an injective function $g: S \to \{0,1\}^n$ such that $Fg = gf$.

Consider, then, the following synthesis problem : For a given function $f : [n] \to [n]$ where n is an arbitrary integer, construct a simple e-r NN $N(n, \mathbf{ba}, \theta)$ whose state transition function F_N realizes f.

First, assume that the following holds for a positive number ε.

$$\frac{\theta_0}{b_0} < \frac{\theta_1}{b_1} < \frac{\theta_2}{b_2} < \cdots \cdots < \frac{\theta_n}{b_n} < \frac{\theta_{n+1}}{b_{n+1}} \quad \text{such that} \quad \frac{\theta_{i+1}}{b_{i+1}} - \frac{\theta_i}{b_i} \geq 2\varepsilon \quad \text{for i in [n]} \quad (3.4)$$

where we put, for notational convention, $\dfrac{\theta_0}{b_0} \leq \dfrac{\theta_1}{b_1} - 2\varepsilon$ and $\dfrac{\theta_{n+1}}{b_{n+1}} \geq \dfrac{\theta_n}{b_n} + 2\varepsilon$.

If we assign a standard state $s_k = (1,1,\ldots,1,0,0,\ldots,0)$ to an integer k in [n], it is enough to have

$$\frac{\theta_{f(k)}}{b_{f(k)}} < A_k < \frac{\theta_{f(k)+1}}{b_{f(k)+1}} \qquad \text{where } A_k = \sum_{j=1}^{k} a_j \quad \text{for every k in [n].} \quad (3.5)$$

This is possible, for example, if we let $A_k = \dfrac{\theta_{f(k)}}{b_{f(k)}} + \varepsilon$, that is, if

$$a_k = A_k - A_{k-1} = \frac{\theta_{f(k)}}{b_{f(k)}} - \frac{\theta_{f(k-1)}}{b_{f(k-1)}} \quad \text{for every k in } [n]-\{0\}. \quad (3.6)$$

Since $A_0 = 0$ by definition, we should have

$$\frac{\theta_{f(0)}}{b_{f(0)}} + \varepsilon = 0. \quad (3.7)$$

Since we can determine the parameters $\mathbf{a}$, $\mathbf{b}$, and θ for a simple e-r NN $N(n, \mathbf{ba}, \theta)$ that satisfy the conditions (3.4), (3.6), and (3.7) then we have the global function F_N which realizes f

through the correspondence $g : [n] \rightarrow S_{st}$ such that $g(k) = s_k$ for k in [n]. Now the synthesis problem posed here is solved and we have

Theorem 3.4.

Given an arbitrary function $f : [n] \rightarrow [n]$, we have a simple full e-r NN with n neurons whose global function F realizes f.

3.3. Analysis of Full e-r NN

In this subsection, we generalize the analyses for simple full e-r NN in Section 3.1. a little bit as follows. That is, we consider the behavior of a full e-r NN $N(n, \mathbf{ba}, \theta)$ without the simpleness restriction. Then there is a simple e-r NN $N_s(n, \mathbf{b_s a_s}, \theta_s)$ which accompanies with $N(n, \mathbf{ba}, \theta)$ such that $\mathbf{a_s} = \mathbf{a}$ and $\theta_s/\mathbf{b_s} = \theta/\mathbf{b}$ where the equality means that of each corresponding component. That is, $b_{s_i} = -b_i$ and $\theta_{s_i} = -\theta_i$ if $b_i < 0$, and $b_{s_i} = b_i$ and $\theta_{s_i} = \theta_i$ if $b_i > 0$.

Now, we define a function which changes a state at neuron i where $b_i < 0$:

Let $h_i : \{0,1\} \rightarrow \{0,1\}$ be defined by

$$h_i(x) = \begin{cases} x & \text{if } b_i > 0 \\ \bar{x} & \text{if } b_i < 0. \end{cases}$$

Then we can define the product $h = \prod_{i=1}^{n} h_i : \{0,1\}^n \rightarrow \{0,1\}^n$ such that

$$h(s_1, s_2,...,s_n) = (h_1(s_1), h_2(s_2),...,h_n(s_n)).$$

For a full e-r NN $N(n, \mathbf{ba}, \theta)$, we have

$$s_i(t+1) = H(b_i\{\sum_{j=1}^{n} a_j s_j(t) - \frac{\theta_i}{b_i}\}) \qquad (i=1,2,...,n) \qquad (3.8)$$

If $F_{N_s}(s) = s'$ then $F_N(s) = h(s')$. That is, we have

Lemma 3.5.

$F_N(s) = h(F_{N_s}(s))$ for any s in S.

Note that since h is bijective such that $h^{-1} = h$, we also have $F_{N_s}(s) = h(F_N(s))$. Although these relations do not necessarily imply an isomorphism between F_N and F_{N_s}, the global state transition structures are similarly characterized as shown below.

We define standard states for an e-r NN $N(n, \mathbf{ba}, \theta)$ through those for the accompanying simple e-r NN $N_s(n, \mathbf{b_s a_s}, \theta_s)$: $\widehat{s_i}$ is a standard state for $N(n, \mathbf{ba}, \theta)$ if it is equal to $h(s_i)$ for a standard state s_i of $N_s(n, \mathbf{b_s a_s}, \theta_s)$. Let $\widehat{S_{st}}$ denote the set of standard states for $N(n, \mathbf{ba}, \theta)$ i.e., $\widehat{S_{st}} = \{ \widehat{s_i} \mid \widehat{s_i} = h(s_i)$ for $i = 0,1,2,...,n\}$

Theorem 3.6.

Let $N(n, \mathbf{ba}, \theta)$ be an e-r NN with normal ordering where the global state transition is denoted by F_N. Let S and $\widehat{S_{st}}$ be the sets of all states and standard states, respectively. Then we have

$$\widehat{S_{st}} \supset F_N(S) .$$

In particular, $\widehat{S_{st}} \supset F_N(\widehat{S_{st}})$.

The theorem can be proved by noting the relation $\widehat{S_{st}} = h(S_{st}) \supset h(F_{N_s}(S)) = F_N(S)$. A similar consideration as for Theorem 3.2. leads us to the same conclusion for the full e-r NN as in Lemma 3.3.

Lemma 3.7.

For the global state transitions of a full e-r NN with n neurons:

(1) Cyclic states and in particular, stable states, if any, are composed of the standard states. Thus the number of states which belong to one of the cycles is at most n+1.

(2) The maximum cycle length is at most n+1.

(3) The number of stable states is at most n+1.

(4) The length of the longest path (which is composed of one non-standard state and some of the standard states) to a cyclic state is at most n+1.

4. Quasi-Symmetric e-r NN

In this section, we continue the discussion for the quasi-symmetric NN and show that a certain class of e-r NNs is a special case of quasi-symmetric NNs and hence has Lyapunov functions both for the asynchronous and synchronous operation.

We have shown in Section 2 that a quasi-symmetric NN $N(n, W, \theta)$ where $w_{ij} = v_{ij}c_j$ such that $v_{ij} = v_{ji}$, $c_j > 0$, and $w_{ii} = 0$ for i j = 1,2,...,n has the following Lyapunov function under asynchronous operation mode.

$$F_a(t) = - \sum_{i,j=1}^{n} c_i w_{ij} s_i(t) s_j(t) + 2\sum_{i=1}^{n} c_i s_i(t)\theta_i \tag{4.1}$$

A similar generalization of Goles' result is possible for any quasi-symmetric NN where $w_{ij} = v_{ij}c_j$ such that $v_{ij} = v_{ji}$, $c_j > 0$ for i,j = 1,2,...,n under synchronous operation mode. Consider the following function $F_s(t)$.

$$F_s(t) = - \sum_{i,j=1}^{n} c_i w_{ij} s_i(t) s_j(t-1) + \sum_{i=1}^{n} c_i(s_i(t) + s_i(t-1))\theta_i \tag{4.2}$$

By simple calculations, we have

$$F_s(t) - F_s(t-1) = - \sum_{i=1}^{n} c_i d_i(t-1)(s_i(t) - s_i(t-2))$$

$$\text{where } d_i(t-1) = \sum_{j=1}^{n} w_{ij} s_j(t-1) - \theta_i \tag{4.3}$$

If $d_i(t-1) > 0$ then $s_i(t) = 1$ which means $s_i(t) - s_i(t-2) \geq 0$. If $d_i(t-1) < 0$ then $s_i(t) = 0$ which means $s_i(t) - s_i(t-2) \leq 0$. In both cases, we have $c_i d_i(t-1)(s_i(t) - s_i(t-2)) \geq 0$ because c_i is positive. Thus $F_s(t)$ is a monotone non-increasing function of t and we have

Theorem 4.1.

Let $N(n, W, \theta)$ be a quasi-symmetric NN where $w_{ij} = v_{ij}c_j$ such that $v_{ij} = v_{ji}$ and $c_j > 0$ for $i,j = 1,2,...,n$. Then the cycle lengths of the global state transition F_N are at most two.

The weight parameter of an e-r NN $N(n, \mathbf{ba}\bullet C, \theta)$ is given by $w_{ij} = b_i a_j c_{ij}$ as defined in Section 3. If we assume symmetric connection (i.e., $c_{ij} = c_{ji}$), then we have quasi-symmetric systems by putting $v_{ij} = b_i b_j c_{ij}$ and $c_j = a_j/b_j$ for $i,j = 1,2,...,n$. Then by Theorems 2.2. and 4.1, we have the following results for the global behaviors of quasi-symmetric e-r NNs.

Theorem 4.2.

Let $N(n, \mathbf{ba}\bullet C, \theta)$ be an e-r NN such that C is a symmetric matrix. If $a_j b_j > 0$ for $j = 1,2,...,n$, then the cycle lengths of the global state transition for synchronous operation mode are at most two. If in addition, $c_{ii} = 0$ for $i = 1,2,...,n$, then any state configuration approaches to some stable state under asynchronous operation mode.

5. Concluding Remarks

We defined a class of asymmetric neural networks characterized by effector and receptor parameters and revealed basic structures of global state transitions for synchronous and asynchronous modes of operation. Although the class seems to be rather restrictive especially when we assume complete connection, it has been shown that any state transition can be realized by a network in this class under synchronous operation. Further, any McCulloch-Pitts type neural network may be regarded as the one with generalized effector and receptor parameters, and investigation in this direction is now under progress.

The analyses of possible forms of energy functions for asynchronously operating neural networks done in this paper are also relevant when we want to speed up the network operation. That is, the formulas for the energy difference given here enable one to decide easily when it is possible to carry out (partially) parallel state transition keeping an energy function decreasing.

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