Notes on Imperfect Repair by Fumio OHI Osaka University

0. Introduction

A component with failure time distribution \( F(t) = 1 - e^{-\Lambda(t)} \) is repaired at failure. Each repair results in minimal repair or perfect repair. Let \( N \) be a positive integer valued random variable denoting the number of repairs performed till the perfect repair, i.e., \( N = k \) means the event that \( k-1 \) minimal repairs were performed and the repair for the \( k \)-th failure was perfect and the component returned to the good-as-new-state. After the perfect repair the process is renewed. Time for repair is assumed to be negligible.

The dynamic process of the component is governed by a non-homogeneous Poisson process \( \{N(t), \ t \geq 0\} \) with mean value function \( \Lambda(t) \). Then \( T_N \) means the time that the component returns to the good-as-new-state, where \( T_k = \inf \{t | N(t) = k\}, k \geq 1 \), the time that the \( k \)-th failure occurs supposed that the repairs for the previous \( k-1 \) failures were minimal.

In this paper we study monotonic properties of \( T_N \) and other stochastic quantities, e.g., steady-state-distributions and so on. Our results may be of interest in renewal theory as well as in reliability theory.

1. Preliminaries

\( \{N(t), t \geq 0\} \) : a non-homogeneous Poisson Process with differentiable mean value function \( \Lambda(t) \),

\[
\Pr[N(t) = k] = e^{-\Lambda(t)} \frac{[\Lambda(t)]^k}{k!},
\]

\( \lambda(t) = \frac{d}{dt} \Lambda(t) \),
\[ T_k = \inf \{ t \mid N(t) = k \}, \quad k = 1, 2, \ldots, \]
\[ T_0 \equiv 0. \]

The following theorem is easily proved.

**Theorem 1.** For \( k \geq 0, \; \ell \geq 1, \)

1. \[ \Pr[T_{k+\ell} - T_k > x \mid T_k = y] = \Pr[N(x+y) - N(y) < y], \]
2. \[ \Pr[T_{k+\ell} - T_k > x] = \int_0^\infty \Pr[N(x+y) - N(y) < \ell] \Pr[T_k \leq y] \, dy, \]
3. \[ E[T_{k+\ell} - T_k \mid T_k = y] = \int_0^y \Pr[N(x+y) - N(y) < \ell] \, dx, \]
4. \[ E[T_{k+\ell} - T_k] = \int_0^\infty \Pr[N(x+y) - N(y) < \ell] \Pr[T_k \leq y] \, dx. \hspace{1cm} \Box \]

**Corollary 2.** Letting \( \ell = 1 \) in the previous theorem, for \( k \geq 1, \)

1. \[ \Pr[T_{k+1} - T_k > x \mid T_k = y] = e^{-[A(x+y) - A(y)]} = \frac{e^{-A(x+y)}}{e^{-A(y)}}, \]
2. \[ \Pr[T_{k+1} - T_k > x] = \int_0^\infty \frac{e^{-A(x+y)}}{e^{-A(y)}} \Pr[T_k \leq y] \, dy, \]
3. \[ E[T_{k+1} - T_k \mid T_k = y] = \int_0^y \frac{e^{-A(x+y)}}{e^{-A(y)}} \, dx, \]
4. \[ E[T_{k+1} - T_k] = \int_0^\infty \frac{e^{-A(x+y)}}{e^{-A(y)}} \Pr[T_k \leq y] \, dx. \hspace{1cm} \Box \]

In the sequel we use the following lemmas, of which proof is easy and then omitted.

**Lemma 3.**

Let \( g(x) \uparrow, \; g(x) \geq 0 \) for \( x \geq 0, \; f(x) \uparrow, \; f(x) \geq 0 \) for \( x \geq 0. \) If two distribution functions \( F_1 \) and \( F_2 \) satisfy \( F_1(0^-) = F_2(0^-) = 0 \) and \( F_1(x) \leq F_2(x) \) for \( x \geq 0, \) then

\[ \int_0^\infty g(x) \, dF_1(x) < \int_0^\infty g(x) \, dF_2(x) \quad \text{and} \quad \int_0^\infty f(x) \, dF_1(x) > \int_0^\infty f(x) \, dF_2(x), \]

supposing that the integrations finitely exist. \hspace{1cm} \Box

**Lemma 4.** For \( \lambda > \mu, \; \sum_{k=0}^n e^{-\lambda} \frac{\lambda^k}{k!} < \sum_{k=0}^n e^{-\mu} \frac{\mu^k}{k!} \) holds for \( \forall n \geq 0. \) \hspace{1cm} \Box

**Theorem 5.** For \( k \geq 0, \; \ell \geq 1, \)

1. \[ 1-e^{-t} : \text{IFR} \Rightarrow \Pr[T_{k+\ell} - T_k > x] \uparrow_k \quad \text{for} \; \forall x \geq 0, \]

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(2) $1 - e^{-\Lambda(t)} : DMRL \Rightarrow E[T_{k+\xi - T_k}] \leq 1_k$

(3) $1 - e^{-\Lambda(t)} : NBU \Rightarrow \Pr[T_{k+\xi - T_k} > x] \leq \Pr[T_{\xi} > x]$ for $\forall x > 0$,

(4) $1 - e^{-\Lambda(t)} : NBUE \Rightarrow E[T_{k+\xi - T_k}] \leq E[T_1]$.

Proof. We notice that $T_{k+\xi} \uparrow k$ a.s.

(1) $1 - e^{-\Lambda(t)} : IFR \Leftrightarrow A(x+y) - A(y) \tau_y \Rightarrow \Pr[N(x+y) - N(y) < \xi] \downarrow y$ (by Lemma 4)

$\Rightarrow \Pr[T_{k+\xi - T_k} > x] \downarrow k$ (by Theorem 1 (2) and Lemma 3).

(2) $E[T_{k+\xi - T_k}] = \sum_{i=1}^{k-1} E[T_{k+i+1} - T_{k+i}]$ is decreasing in $k$, since each term of the right hand side is decreasing in $k$ by Corollary 2 and Lemma 3.

(3) By Lemma 4, $\Pr[N(x+y) - N(y) < \xi] \leq \Pr[N(x) < \xi] = \Pr[T_{\xi} > x]$. Then (3) is obvious.

(4) $E[T_{k+\xi - T_k}] = \sum_{i=0}^{k-1} E[T_{k+i+1} - T_{k+i}]$ and $E[T_{k+i+1} - T_{k+i}] \leq \int_0^{\infty} e^{-\Lambda(x)} dx = E[T_1]$ by the assumption and Corollary 2 (4). Then we have the inequality.

\[ \Box \]

2. Monotonic Properties of $T_k$

Let $N$ be a positive integer valued r.v. independent with $\{N(t), t \geq 0\}$. In this section we study monotonic properties of $T_N$.

Theorem 6. (1) Suppose that $Z_i$ (i.i.d.) are i.i.d. r.v.'s with common distribution function same to the one of $T_1$, and are independent with $N$.

(1) $1 - e^{-\Lambda(t)} : NBU \Rightarrow \Pr[T_N > t] \leq \Pr[\sum_{i=1}^{N} Z_i > t]$.

(2) $1 - e^{-\Lambda(t)} : NBUE \Rightarrow E[T_N] \leq E[N]E[T_1]$.

Proof. (2) is obvious from Theorem 5 (4).

(1) It is sufficient to prove $\Pr[T_k > t] \leq \Pr[Z_1 + \ldots + Z_k > t]$ for $k \geq 1$. We prove the inequality by the mathematical induction on $k$.

\[ \Pr[T_{k+1} > t] = \int \Pr[T_{k+1} > t | T_k = x] d\Pr[T_k = x] \]

\[ = \int \Pr[T_{k+1} - T_k > t-x | T_k = x] d\Pr[T_k = x] \]

\[ \leq \int \Pr[Z_{k+1} > t-x] d\Pr[T_k = x] \quad \text{(by Corollary 2 (1) and th} \]
assumption)
\[
\leq \int \Pr[Z_{k+1} > t-x] d\Pr[Z_1 + \ldots + Z_k \leq x] \quad \text{(by the inductive assumption and Lemma 3)}
\]
\[
= \Pr[Z_1 + \ldots + Z_{k+1} > t]. \quad \square
\]

3. Stochastic Comparisons of $T_N$ and $T_{N'}$

Let
\[
\tilde{F}_N(t) = \Pr[T_N > t] = \sum_{k=0}^{\infty} \Pr[N(t) = k] \Pr[N > k],
\]
\[
f_N(t) = \frac{d}{dt} \Pr[T_N \leq t] = \sum_{k=0}^{\infty} \Pr[N(t) = k] \lambda(t) \Pr[N = k+1],
\]
\[
\lambda_N(t) = \frac{f_N(t)}{\tilde{F}_N(t)}.
\]

In this section $N$ and $N'$ are positive integer valued r.v.'s independent with $\{N(t), t \geq 0\}$

**Theorem 7.** (1) \[ \frac{\Pr[N = \ell]}{\Pr[N = k]} \leq \frac{\Pr[N' = \ell]}{\Pr[N' = k]} \quad \text{for } k < \ell \Rightarrow \frac{f_N(x+\Delta)}{\tilde{F}_N(x)} \leq \frac{f_{N'}(x+\Delta)}{\tilde{F}_{N'}(x)} \quad \text{for } x \geq 0 \text{ and } \Delta > 0. \]

(2) \[ \frac{\Pr[N = k+1]}{\Pr[N > k]} \leq \frac{\Pr[N' = k+1]}{\Pr[N' > k]} \quad \text{for } k \geq 1 \Rightarrow \frac{\Pr[N > \ell]}{\Pr[N > k]} \geq \frac{\Pr[N' > \ell]}{\Pr[N' > k]} \quad \text{for } k < \ell \]
\[ \Rightarrow \frac{\tilde{F}_N(t+\Delta)}{\tilde{F}_N(t)} \geq \frac{\tilde{F}_{N'}(t+\Delta)}{\tilde{F}_{N'}(t)} \quad \text{for } t \geq 0, \Delta > 0 \Rightarrow \lambda_N(t) \leq \lambda_{N'}(t) \quad \text{for } t \geq 0. \]

(3) \[ \Pr[N > \ell] \geq \Pr[N' > \ell] \quad \text{for } t \geq 1 \Rightarrow \tilde{F}_N(t) \geq \tilde{F}_{N'}(t) \quad \text{for } t \geq 0. \]

**Proof.** (1)
\[
\sum_{k=0}^{\infty} \Pr[N(x+\Delta) = k] \lambda(x+\Delta) \Pr[N = k+1] \quad \sum_{k=0}^{\infty} \Pr[N(x+\Delta) = k] \lambda(x+\Delta) \Pr[N' = k+1]
\]
\[
\sum_{k=0}^{\infty} \Pr[N(x) = k] \lambda(x) \Pr[N = k+1] \quad \sum_{k=0}^{\infty} \Pr[N(x) = k] \lambda(x) \Pr[N' = k+1]
\]
\[
= \sum_{k < \ell} \Pr[N(x+\Delta) = k] \lambda(x+\Delta) \Pr[N(x+\Delta) = \ell] \lambda(x) \Pr[N = k+1] \Pr[N' = k+1] \quad \Pr[N = \ell+1] \Pr[N' = \ell+1] \leq 0.
\]

(2) The equivalent relations of (2) is obvious.
\[
\left| \frac{\tilde{F}_N(t+\Delta)}{\tilde{F}_N(t)} \right| \left| \frac{\tilde{F}_{N'}(t+\Delta)}{\tilde{F}_{N'}(t)} \right| = \sum_{k < \ell} \Pr[N(t+\Delta) = k] \Pr[N(t+\Delta) = \ell] \Pr[N > k] \Pr[N' > k] \]
\[
\left| \frac{\tilde{F}_N(t)}{\tilde{F}_N(t)} \right| \left| \frac{\tilde{F}_{N'}(t)}{\tilde{F}_{N'}(t)} \right| = \sum_{k < \ell} \Pr[N(t) = k] \Pr[N(t) = \ell] \Pr[N > k] \Pr[N' > k] \]
\[
\left| \frac{\tilde{F}_N(t+\Delta)}{\tilde{F}_N(t)} \right| \left| \frac{\tilde{F}_{N'}(t+\Delta)}{\tilde{F}_{N'}(t)} \right| = \sum_{k < \ell} \Pr[N(t) = k] \Pr[N(t) = \ell] \Pr[N > k] \Pr[N' > k] \]
\[
\left| \frac{\tilde{F}_N(t)}{\tilde{F}_N(t)} \right| \left| \frac{\tilde{F}_{N'}(t)}{\tilde{F}_{N'}(t)} \right| = \sum_{k < \ell} \Pr[N(t) = k] \Pr[N(t) = \ell] \Pr[N > k] \Pr[N' > k] \]
\[ \geq 0. \]

The relation (3) is easily proved by using Lemma 3. \( \square \)

We present simple bounds for the distribution and the expectation of \( T_N \).

**Corollary 8.** Let \( q_m = \inf_k \Pr[N > k+1|N > k], \ q_M = \sup_k \Pr[N > k+1|N > k], \) and \( N_m \) and \( N_M \) be positive integer valued r.v.'s independent with \( \{N(t), t \geq 0\} \) such that \( \Pr[N_m = k] = q_m^{k-1}(1-q_m), \ \Pr[N_M = k] = q_M^{k-1}(1-q_M). \) Since \( \Pr[N_m > k] \leq \Pr[N > k] \leq \Pr[N_M > k] \) for \( k \geq 1 \), by Theorem 7 we have \( \Pr[T_{N_m} > t] \leq \Pr[T_N > t] \leq \Pr[T_{N_M} > t] \) for \( t \geq 0 \) and \( E[T_{N_m}] \leq E[T_N] \leq E[T_{N_M}] \). \( \square \)

**Remark 9.** Theorem 7 (1) (2) (3) show that stochastically-larger-relations between \( N \) and \( N' \) are preserved to the same stochastic relations between \( T_N \) and \( T_N' \), without any assumption on \( 1-e^{-\Lambda(t)} \). \( \square \)

**Theorem 10.**

\[ 1-e^{-\Lambda(t)}: \text{DMRL}, \ \sum_{k=1}^{\infty} \frac{\Pr[N \geq k]}{E[N]} \geq \sum_{k=1}^{\infty} \frac{\Pr[N' \geq k]}{E[N']} \text{ for } j \geq 1 \]

\[ \Rightarrow \frac{E[T_N]}{E[N]} \leq \frac{E[T_{N'}]}{E[N']} \text{ .} \]

**Proof.** Since \( 1-e^{-\Lambda(t)} \) is DMRL, \( E[T_{j'-T_{j-1}}] \) is decreasing in \( j \) by Theorem 5 (2). Then using Lemma 3,

\[ \frac{E[T_N]}{E[N]} = \sum_{j=1}^{\infty} E[T_{j'-T_{j-1}}] \frac{\Pr[N \geq j]}{E[N]} \leq \sum_{j=1}^{\infty} E[T_{j'-T_{j-1}}] \frac{\Pr[N' \geq j]}{E[N']} = \frac{E[T_{N'}]}{E[N']} \text{ .} \ \square \]

**Remark 11.** The following relation holds.

\[ \frac{\Pr[N \leq t]}{\Pr[N = k]} \geq \frac{\Pr[N' \leq t]}{\Pr[N' \geq k]} \text{ for } k \leq t \Rightarrow \frac{\Pr[N \leq t]}{\Pr[N = k]} \geq \frac{\Pr[N' \leq t]}{\Pr[N' \geq k]} \text{ for } k \leq t \]

\[ \sum_{k=1}^{\infty} \frac{\Pr[N \geq k]}{E[N]} \geq \sum_{k=1}^{\infty} \frac{\Pr[N' \geq k]}{E[N']} \text{ for } j \geq 1 \text{ .} \ (1) \]

\[ \text{Pr}[N \geq k] \geq \text{Pr}[N' \geq k] \text{ for } k \geq 1 \text{ .} \ (2) \]

There is generally no relation between (1) and (2). \( \square \)

**Remark 12.** It is easily verified that if \( \Pr[N \leq 2] = \Pr[N' \leq 2] = 1, \) \( \Pr[N = 2] \geq \Pr[N' = 2] \) and \( 1-e^{-\Lambda(t)} \) is NBUL, then \( \frac{E[T_N]}{E[N]} \leq \frac{E[T_{N'}]}{E[N']} \text{ .} \ \square \)
Lemma 13. For \( a_1 \geq a_2 \geq \ldots \geq a_n \) holds for \( n \geq 1 \). The proof is easy and omitted. □

Theorem 14. \( 1-e^{-\lambda(t)} \) is DMRL and \( \Pr[N > k] \geq \Pr[N' > k] \) for \( k \geq 1 \)

\[
\mathbb{E}\left[ \frac{T_N}{N-1} \right] \leq \mathbb{E}\left[ \frac{T_{N'}}{N'} \right].
\]

Proof. Since \( 1-e^{-\lambda(t)} \) is DMRL, \( \mathbb{E}[T_j-T_{j-1}] \) is decreasing in \( j \) by Theorem 5 (2). Then by Lemma 13, \( \mathbb{E}[T_k/k] = \frac{\sum_{j=1}^{k} \mathbb{E}[T_j-T_{j-1}]}{k} \) is decreasing in \( k \). Then Theorem 14 is obvious by Lemma 3. □

4. Stochastic Comparisons of Steady-State-Distributions

\( \{N^j(t), t \geq 0\} \) (\( j \geq 1 \)) : independent non-homogeneous Poisson processes with common mean value function \( \lambda(t) \),

\[
T_k = \inf \{ t | N^j(t) = k \}
\]

\( N_j \) (\( j \geq 1 \)) : independent positive integer valued r.v.'s with common distribution same to the one of \( N \),

i.e., \( \{N^j(t), t \geq 0\} \) (\( j \geq 1 \)) are replicas of \( \{N(t), t \geq 0\} \) and \( N_j \) (\( j \geq 1 \)) are replicas of \( N \). We assume that \( \{N^j(t), t \geq 0\} \) (\( j \geq 1 \)) are independent with \( N_j \) (\( j \geq 1 \)).

We define a counting process \( \{M(T), t \geq 0\} \) as

\[
M(t) = \sum_{j=1}^{n-1} N_j + N^n(t-\sum_{j=1}^{n-1} T_j) \quad \text{if} \quad \sum_{j=1}^{n-1} T_j \leq t \leq \sum_{j=1}^{n} T_j.
\]

We notice that \( T_j \) (\( j \geq 1 \)) are i.i.d. random variables with common distribution function \( F_N(t) \), 1-\( F_N(t)=\sum_{k=0}^{\infty} \Pr[N(t)=k]\Pr[N=k] \). In this section we consider stochastic quantities with respect to \( \{M(t), t \geq 0\} \), which means the number of repairs performed in \([0,t]\).

Let's define

\[
Z(t) = \frac{T^n}{N^n(t-\sum_{j=1}^{n-1} T_j)+1} \quad \text{if} \quad \sum_{j=1}^{n-1} T_j \leq t \leq \sum_{j=1}^{n} T_j,
\]

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which means the time to the next failure from the time epoch $t$.

**Theorem 15.**

$$\lim_{t \to \infty} \Pr[Z(t)] = \int_0^\infty \frac{F_N(y)}{E[T_N]} e^{-A(y)} dy .$$

**Proof.** Simple calculation verifies the above equality. Since

$$\Pr[Z(t)] = \int_0^\infty \prod_{n=1}^{\infty} \Pr[T_N(x-y) + 1 > t-y+x, t-y < T_N] \prod_{n=1}^{\infty} (F_N)^{(n-1)}(y) ,$$

where $(F_N)^{(n-1)}$ is the $(n-1)$-hold convolution of $F_N^N$, then by the basic renewal theory we have

$$\lim_{t \to \infty} \Pr[Z(t)] = \frac{1}{E[T_N]} \int_0^\infty \Pr[T_N(y) > t+y, t+y < T_N] dy .$$

Noticing that $\Pr[T_N(y) > t+y, t+y < T_N] = \bar{F}_N(y) \Pr[N(y+y) - N(y) = 0]$, the theorem is proved.

We write the steady-state-distribution of $Z(t)$ as $H_N$, i.e.,

$$H_N(x) = 1 - H_N(x) = \int_0^\infty \frac{F_N(y)}{E[T_N]} e^{-A(y)} dy .$$

**Theorem 16.** (1) The density function $h_N(x)$ and the failure rate function $r_N(x)$ of $H_N(x)$ are

$$h_N(x) = \frac{\int_0^\infty F_N(y) e^{-A(y)} \chi(x+y) dy}{E[T_N]} ,$$

$$r_N(x) = \frac{\int_0^\infty F_N(y) \frac{e^{-A(y)} \chi(x+y)}{e^{-A(y)}} dy}{E[T_N]} .$$

(2) $\Pr[N\geq t] \leq \Pr[N' > t] \leq \Pr[N' > t] \Pr[N > t]$ for $k < t$, $1 - e^{-A(t)}$ has PF$_2$-density

$$\Rightarrow \frac{h_N(t+\Delta)}{h_N(t)} \leq \frac{h_N(t)}{h_N(t+\Delta)} \quad \text{for } t > 0, \Delta > 0 .$$

(3) $\Pr[N\geq t] \leq \Pr[N' > t] \leq \Pr[N' > t] \Pr[N > t]$ for $k < t$, $1 - e^{-A(t)}$ : IFR

$$\Rightarrow r_N(x) \leq r_N'(x) \quad \text{for } \forall x > 0 \Rightarrow H_N(x) \leq H_N'(x) \quad \text{for } \forall x > 0 .$$
**Proof.** Differentiating $H_N(x)$, (1) is easily obtained.

\[
\begin{align*}
(2) & \quad \int_0^\infty \frac{F_N(y)}{e^{-\Lambda(y)}} \cdot \lambda(x+y) dy = \int_0^\infty \frac{F_N(y)}{e^{-\Lambda(y)}} \cdot \lambda(x+y) dy \\
& \quad \int_0^\infty \frac{F_N'(y)}{e^{-\Lambda(y)}} \cdot \lambda(x+y) dy = \int_0^\infty \frac{F_N'(y)}{e^{-\Lambda(y)}} \cdot \lambda(x+y) dy \\
& \quad = \int \int \left[ F_N(y_1) F_N(y_2) \right] \left[ \frac{e^{-\Lambda(x+y_1)}}{e^{-\Lambda(y_1)}} \cdot \lambda(x+y_1) \right] \left[ \frac{e^{-\Lambda(x+y_2)}}{e^{-\Lambda(y_2)}} \cdot \lambda(x+y_2) \right] dy_1 dy_2 \geq 0.
\end{align*}
\]

Using Basic Composition Theorem, (3) is proved similarly to the proof of (2). □

We define

\[ Z^*(t) = \sum_{j=1}^n T_N^j - t \quad \text{if} \quad \sum_{j=1}^{n-1} T_N^j \leq t \leq \sum_{j=1}^n T_N^j , \]

which means the time to the next perfect repair from the time epoch $t$.

It is well known that

\[ \lim_{t \to \infty} \Pr[Z^*(t) \leq x] = \frac{1}{E[T_N]} \int_0^x F_N(u) du . \]

We write the right hand side of the above equality as $H_N^*(x)$.

**Theorem 17.** \[ \Pr[N > \ell] \leq \Pr[N'^{\ell'}] \quad \text{for } k < \ell \]

\[ \Rightarrow \frac{H_N^*(t+\Delta)}{H_N^*(t)} \leq \frac{H_N'^{\ell'}(t+\Delta)}{H_N'^{\ell'}(t)} \quad \text{for } t > 0, \Delta > 0 \Rightarrow H_N^*(t) \geq H_N'^{\ell'}(t) \quad \text{for } t > 0 . \]

**Proof.**

\[
\begin{align*}
& \int_{t+\Delta}^\infty F_N(u) du = \int_{t+\Delta}^\infty F_N(u) du \\
\text{and} & \int_{t}^{t+\Delta} F_N'(u) du = \int_{t}^{t+\Delta} F_N'(u) du \leq 0.
\end{align*}
\]

Noticing that \( H_N(x) = \int_0^x \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} \cdot \lambda(x+y) \) \( dH_N^*(y) \), we have by Theorem 17 and Lemma 3.
Theorem 18. \[ \frac{\Pr[N > t]}{\Pr[N > k]} \leq \frac{\Pr[N' > t]}{\Pr[N' > k]} \] for \( k \leq t \), \( 1 - e^{-\lambda(t)} : \text{DMRL} \)

\[ \Rightarrow \int_0^\infty R_N(x)dx \geq \int_0^\infty R_{N'}(x)dx . \]

References.


