数理解析研究所講究録
第 695 巻 1989 年 85-97

S-bases of Boolean Functions Under Several Functional Constructions - A Survey -

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ABSTRACT

We determine formulas for the numbers of s-bases of Boolean functions (bases consisting solely of symmetric functions) containing only n-ary functions under six kinds of functional constructions besides ordinary composition. This is done on the basis of the classification and basis enumeration results of the functional completeness theory.

1. Introduction

In the synthesis of switching circuits the set of given gates should construct any switching function. Such a set of logical functions is called a *base*. Due to practical reasons elements of a base are usually selected from symmetric functions. Thus, bases consisting of symmetric functions (so called *s-bases*) are of special interest.

The notion of s-base was introduced systematically first in [Tos72] under ordinary composition and respective formulas for the numbers of s-bases consisting solely of n-ary functions and solely of up-to n-ary functions (denoted by N^n and $N^{\leq n}$, respectively) were given there. In the present paper we survey the same topic under known six different ways of composition besides ordinary one and obtain similar formulas N^n (or $N^{\leq n}$) for each composition. This is done according to the following scheme.

Assume that sets of functions, called *classes*, are indexed by i. Define *n-profile* $p_n(i)$ of a class i by the number of n-ary symmetric functions in the set. Let us call the number of functions of a base the *rank* of a base and let N_r^n denote the number of s-bases of rank r consisting solely of n-ary functions. Then N^n is determined in the following way.

1. Using the result of the classification of Boolean functions, we divide the set of all symmetric functions into classes and determine n-profile $p_n(i)$ of each class i.
2. We enumerate all s-bases up-to equivalence, i.e. each s-base class (assume its rank r) is represented by classes of functions as $\{i_1, i_2, \ldots, i_r\}$. Then the number of s-bases consisting of n-ary functions in the s-base class is equal to $p_n(i_1) \times p_n(i_2) \cdots \times p_n(i_r)$.

3. Summing these numbers for all s-base classes for a rank r we obtain corresponding data N_r^n. Finally, $N^n = \sum_{r=1}^{\max r} N_r^n$ equals the number of s-bases consisting solely of n-ary functions. Similarly, $N^{\leq n}$ can be obtained using *up-to profile* of a class which is defined by the sum of the profile up to n, i.e., $\sum_{j=1}^{n} p_j(i)$.

In 2 and 3 we need symbolic calculation (in some case sums of more than a hundred terms), which is done by using the computer algebraic system Reduce [MSTM89].

2. Subsets of Boolean symmetric functions

Let $E = \{0,1\}$ and let P_2 denote the set of all logical functions, i.e. the union of all functions $f : E^n \rightarrow E$ for $n = 1, 2, \ldots$. The operation of superposition (or composition) of functions is defined formally in the following way (cf. [Ros77]): If f, g are m-ary and n-ary functions from a set $F \subseteq P_2$, then each function obtained from f by permuting and identifying variables and the $(m+n-1)$-ary function h defined by setting

$$h(x_1, \ldots, x_{m+n-1}) := f(g(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{m+n-1})$$

is a *superposition.* Note that neither delays nor loop-structure are allowed for the composition; we will consider situations where several conditions are posed on the composition in Sections 3, 4 and 5. So this composition we refer as *ordinary* one.

From the functional completeness theory we know that we can partition P_2 into equivalence classes and can discuss "bases" in terms of these classes instead of individual functions. We also know that the classes coincide with non-empty intersections of all maximal sets. Such a class is conveniently represented by a binary m-string (m the number of maximal sets), called a *characteristic vector,* $a_1 \ldots a_m$, where $a_i = 0$ if $f \in H_i$ and $a_i = 1$ otherwise, for maximal sets H_i, $1 \leq i \leq m$. All functions f having the same characteristic vector form a *class of functions.* All bases with the same set of classes form a *class of bases* [Miy88].

A function $f(x_1, \ldots, x_n)$ is said to be *symmetric* if $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ holds for all $x_1, \ldots, x_n \in E$ and every permutation π on $\{1, \ldots, n\}$. Let us denote the so-called *fundamental symmetric function* by s_r^n, which takes the value 1 if and only if its r out of all n arguments assume the value 1. For given n, there exist exactly $n+1$ fundamental symmetric functions: $s_0^n, s_1^n, \ldots, s_n^n$. Each symmetric function can be uniquely represented as a disjunction of fundamental symmetric functions [Sha49]. This gives a notation for symmetric functions, setting ($n \geq 1$): $s_{r_1,\ldots,r_l}^n := s_{r_1}^n \vee \ldots \vee s_{r_l}^n$. For example, the n-ary constant-valued functions c_0^n and c_1^n are symmetric functions, which correspond to s_ϕ^n and $s_{0,1,\ldots,n}^n$, respectively. Hence, the number of n-ary symmetric functions in P_2 is 2^{n+1}, and the number of up-to n-ary symmetric functions is $2^{n+2} - 4$. Let $R = \{r_1, \ldots, r_l\}$ and let $s_R^n = s_{r_1,\ldots,r_l}^n$. We assume further that $0 \leq r_1 < \ldots < r_l \leq n$. Let $x_1 + x_2 + \ldots + x_n$ denote the number of 1's in the vector $(x_1, \ldots, x_n)$. Thus $s_R^n(x_1, \ldots, x_n) = 1 \Leftrightarrow x_1 + \ldots + x_n \in R$.

We denote by H^s the set of symmetric functions from $H \subseteq P_2$. Also we denote the intersection of the sets $H_1, \ldots, H_i$ by $H_1 \ldots H_i$ and the complement set of H by $\overline{H}$, i.e., $\overline{H} = P_2 \setminus H$. Hereafter, $x + y \pmod 2$ and $xy \pmod 2$ are denoted by $x + y$ and xy, respectively.

Definition 2.1. *(see e.g. [MSHMF88])*

1) Functions preserving zero: $T_0 = \{f | f(0, \ldots, 0) = 0\}$.
   $T_0^s = \{s_R^n | 0 \notin R\}$. $|T_0^s(n)| = 2^n$.
2) Functions preserving one: $T_1 = \{f | f(1, \ldots, 1) = 1\}$.
   $T_1^s = \{s_R^n | n \in R\}$. $|T_1^s(n)| = 2^n$.
3) Monotone increasing functions: $M = \{f | f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$ if $x_i \leq y_i$ for all $i\}$.
   $M^s = \{c_0^n = s_\phi^n, s_n^n, s_{n-1,n}^n, \ldots, s_{1,2,\ldots,n}^n, c_1^n = s_{0,1,\ldots,n}^n\}$. $|M^s(n)| = n + 2$.
4) Selfdual functions: $S = \{f | \overline{f(x_1, \ldots, x_n)} = f(\overline{x_1}, \ldots, \overline{x_n})\}$,
   $S^s = \{s_R^n | i \in R$ if and only if $n - i \notin R$ for all $i = 0, \ldots, (n-1)/2$, n odd$\}$.
   $|S^s(n)| = 2^{(n+1)/2}$ for n odd and 0 for n even.
5) Linear functions: $L = \{f | f(x) = a_0 + a_1 x_1 + \ldots + a_n x_n$ for some $a_i \in E\}$.
   $L^s = \{c_0^n, c_1^n, x_1 + \ldots + x_n, 1 + x_1 + \ldots + x_n\}$, $|L^s(n)| = 4$.
6) Conjunctions: $C = [\{c_0^n, c_1^n, \wedge \text{ (conjunction)}\}]$.
   $C^s = \{c_0^n, c_1^n, s_n^n = x_1 \ldots x_n\}$. $|C^s(n)| = 3$.
7) Disjunctions: $D = [\{c_0^n, c_1^n, \vee \text{ (disjunction)}\}]$.
   $D^s = \{c_0^n, c_1^n, s_{1,2,\ldots,n}^n = x_1 \vee \ldots \vee x_n\}$. $|D^s(n)| = 3$.
8) 1-clique functions (1-side intersecting functions): [PMN88]
   $N_0 = \{f |$ if $f(x) = f(y) = 1$ then $x_i = y_i = 1$ for some $i\}$.
   $N_0^s = \{s_R^n | 2r_1 > n$, where r_1 is the smallest in $R\}$.
   $|N_0^s(n)| = 2^{n/2}$ for n even and $2^{(n+1)/2}$ for n odd.
9) 0-clique functions (0-side intersecting functions):
   $N_1 = \{f |$ if $f(x) = f(y) = 0$ then $x_i = y_i = 0$ for some $i\}$.
   $N_1^s = \{s_R^n | 2r < n$, where r is the greatest in $\{0, 1, \ldots, n\} \setminus R\}$.
   $|N_1^s(n)| = 2^{n/2}$ for n even and $2^{(n+1)/2}$ for n odd.
10) Functions exchanging zero and one: $X = \{f | f(x, \ldots, x) = \overline{x}\}$.
   $X^s = \{s_R^n | 0 \in R, n \notin R\}$. $|X^s(n)| = 2^{n-1}$.
11) Monotone decreasing functions: $M' = \{f | f(x) \geq f(y)$ if $x_i \leq y_i$ for all $i\}$.
   $M'^s = \{c_0^n, s_0^n, \ldots, s_{0,1,\ldots,n-1}^n, c_1^n\}$. $|M'^s(n)| = n + 2$.
12) Functions uniting zero and one:
   $K = \{f | f(0, \ldots, 0) = f(1, \ldots, 1)\}$.
   $K^s = \{s_R^n | 0, n \in R$ or $0, n \notin R\}$. $|K^s(n)| = 2^n$.

Note that there exist no symmetric functions in S for n even [ArH63] (this is the reason why our formulas have two different forms depending on whether n is even or odd).

Lemma 2.1. $M^s(n) \subseteq N_0^s \cup N_1^s$.

Lemma 2.2. *For $n \geq 2$, $L^s(n) \cap M^s(n) = L^s(n) \cap M'^s(n) = L^s(n) \cap C^s(n) = L^s(n) \cap D^s(n)$ $= C^s(n) \cap D^s(n) = M^s(n) \cap M'^s(n) = \{c_0^n, c_1^n\}$.*
And
$M'^s(n) \cap N_0^s(n) = L^s(n) \cap N_0^s(n) = c_0^n$, $M'^s(n) \cap N_1^s(n) = L^s(n) \cap N_1^s(n) = c_1^n$.

Corollary 2.1. $L^s(n) \setminus \{c_0^n, c_1^n\} \subseteq (S^s \cup \overline{S}^s)\overline{N_0}^s \overline{N_1}^s \overline{M}^s \overline{M'}^s$.

Lemma 2.3. *For n even $S^s(n) = \phi$. For n odd, $S^s(n) \cap L^s(n) = \{a + x_1 + \ldots + x_n | a = 0, 1\}$,*
$S^s(n) \cap M^s(n) = S^s(n) \cap N_0^s(n) = S^s(n) \cap N_1^s(n) = N_0^s(n) \cap N_1^s(n)$
$= S^s(n) \cap M^s(n) \cap N_0^s(n) \cap N_1^s(n) = \{s_{(n+1)/2,\ldots,n}^n\}$,
and $S^s(n) \cap M'^s(n) = \{s_{0,1,\ldots,(n-1)/2}^n\}$.

Lemma 2.4.
$N_0^s(n) \cap M^s(n) = \{x \ (n = 1), c_0^n, s_{m,m+1,\ldots,n}^n | m > n/2\}$,
$N_1^s(n) \cap M^s(n) = \{x \ (n = 1), c_1^n, s_{m,m+1,\ldots,n}^n | m < n/2 + 1\}$.

Table 1: Profile of the classes of symmetric functions under ordinary composition.

classes	T_0T_1SLM	$n=1$	n even	n odd
1	1 1 1 1 1	0	2^{n-1}	$2^{n-1}-2^{(n-1)/2}$
2	1 1 0 1 1	0	0	$2^{(n-1)/2}-1$
3,4	0 1 1 1 1 (1 0 1 1 1)	0	$2^{n-1}-2$	$2^{n-1}-1$
5	1 1 0 0 1	1	0	1
6,7	1 0 1 0 1 (0 1 1 0 1)	0	1	0
8	0 0 1 1 1	0	$2^{n-1}-n$	$2^{n-1}-2^{(n-1)/2}-n+1$
9,10	1 0 1 0 0 (0 1 1 0 0)	1	1	1
11	0 0 1 1 0	0	n	$n-1$
12	0 0 0 1 1	0	0	$2^{(n-1)/2}-2$
13,14	0 0 0 1 0 (0 0 0 0 1)	0	0	1
15	0 0 0 0 0	1	0	0

Corollary 2.2. $N_0N_1M = \begin{cases} \phi & n \text{ even}, \\ \{x\ (n=1), s^n_{(n+1)/2,(n+1)/2+1,\ldots,n}\} & n \text{ odd}. \end{cases}$

Lemma 2.5. $\begin{array}{l} N_0^s(n) \cap C^s(n) = \{c_0^n, s_n^n = x_1 \wedge \cdots \wedge x_n\}, \\ N_1^s(n) \cap D^s(n) = \{c_1^n, s^n_{1,\ldots,n} = x_1 \vee \cdots \vee x_n\}. \end{array}$

The following theorem on the completeness under ordinary composition and the classification of Boolean functions are well-known.

Theorem 2.1. [Pos21] *P_2 has exactly the following 5 maximal sets under ordinary composition: T_0, T_1, S, L and M.*

The 15 classes of functions and 42 classes of bases of P_2 are well-known [Jab52, INN63, Krn65]. It is also well-known that each from these classes contains symmetric functions, and hence classes of s-bases coincide with classes of bases. We list the profile of each class from [Tos72] which will be used in Section 4. For example, the functions in class 3 are: $s^n_{0,r_1,\ldots,r_l}$, $n>2$ except the constant function c_1^n. The function $x_1+\ldots+x_n+1$ is also excluded for n even.

The number N^n of s-bases of P_2 under ordinary composition consisting of n-ary ($n>1$) functions is $2^n+4^{n-1}-n-4$ if n is even and $2^{(n-1)/2}+4^{n-1}+3\cdot 8^{(n-1)/2}+2^{n-1}-6$ otherwise [Tos72]. The similar formulas for $N^{\leq n}$ are also given there.

3. S-bases under r-line and 2-line fixed codings

r-line coding

Freivalds [Fre68] introduced the notion of completeness under r-line coding (which he called up to coding completeness). In this construction every primary input and primary output of a network consists of "r-lines" and signals 0 or 1 are feeded to each input or taken out from the output as a binary code word with the length r. Two completeness notions were introduced there: one under r-line coding and the other under fixed r-line coding.

Lemma 3.1. [Fre68] *There are 3 maximal sets under r-line coding: L, C and D.*

The classes of P_2 under r-line coding are given in [MIS85]. There is a symmetric representative in each class. Thus, classes of s-bases coincide with those of bases.

Theorem 3.1. *There exist exactly 5 classes of symmetric functions and 4 classes of s-bases under r-line coding.*

The classes and their profiles are shown in Tables 2, 3, respectively. The classes of s-bases are: rank 1: (5); rank 2 : (2,3),(2,4), (3,4).

Table 2: Classes of symmetric functions under r-line coding completeness.

	LCD	n-ary symmetric functions
1	0 0 0	c_0^n, c_1^n, x.
2	0 1 1	$a + x_1 + \ldots + x_n, a = 0$ or 1, for $n > 1; 1 + x$ for $n = 1$.
3	1 0 1	$x_1 \ldots x_n, n > 1$.
4	1 1 0	$x_1 \vee x_2 \ldots \vee x_n, n > 1$.
5	1 1 1	all remaining symmetric functions

Table 3: Profiles and up-to profiles of the classes under r-line coding.

	$n = 1$	$n > 1$
1	3	2
2	1	2
3,4	0	1
5	0	$2^{n+1} - 6$
sum	4	2^{n+1}

	up to n
1	$2n + 1$
2	$2n - 1$
3,4	$n - 1$
5	$2^{n+2} - 6n - 2$
sum	$2^{n+2} - 4$

Theorem 3.2. *The number of s-bases consisting only of n-ary functions ($n \geq 2$) under r-line coding is: $N_1^n = 2^{n+1} - 6$ (Sheffer symmetric functions), $N_2^n = p_n(2)p_n(3) + p_n(2)p_n(4) + p_n(3)p_n(4) = 2 \cdot 1 + 2 \cdot 1 + 1 \cdot 1 = 5$. Thus there are $N^n = 2^{n+1} - 1$ s-bases. Similarly from Table 3 we have the number $N^{\leq n}$ of the s-bases consisting of up-to n-ary functions: $N^{\leq n} = 2^{n+2} + 5n^2 - 14n + 1$.*

2-line fixed coding

The completeness problem under a 2-line fixed coding: $0 \to 01$ and $1 \to 10$ (this is so called "double rail logic") was solved by Ibuki [Ibu68]. Karunanithi and Friedman [KaF78] also considered this completeness independently. This notion is also related with SP-algebra considered in [Cej69].

Lemma 3.2. *There are 6 maximal sets under 2-line fixed coding: N_0, N_1, S, L, C and D.*

The following 12 classes is due to [Ibu68]. There is a symmetric representative in each class. Hence classes of s-bases and classes of bases coincide under 2-line fixed coding.

Theorem 3.3. *There are exactly 12 classes of symmetric functions of P_2 under 2-line fixed coding (Table 4). Their profiles are given in Tables 5, 6.*

Table 4: Classes of symmetric functions under 2-line fixed coding.

	N_0N_1SLDC	n-ary symmetric functions
1	111111	omitted.
2	111011	$a + x_1 + \ldots + x_n,\ n = 2m.$
3	110111	omitted.
4	011111	omitted.
5	101111	omitted.
6	110011	$a + x_1 + \ldots + x_n,\ n = 2m+1,\ 1 + x.$
7	011110	$x_1x_2\ldots x_n,\ n > 1.$
8	101101	$x_1 \vee x_2 \vee \ldots \vee x_n,\ n > 1.$
9	000111	$s^n_{(n+1)/2,\ldots,n},\ n > 1$ odd.
10	101000	$c^n_1.$
11	011000	$c^n_0.$
12	000000	$x.$

There are 28 classes of bases under 2-line fixed coding [Ibu68]: 1 of rank 1, 22 of rank 2 and 5 of rank 3.

Theorem 3.4. *The formula N^n for the number of s-bases under 2-line fixed coding is given in Table 7.*

4. S-bases under compositions with delayed functions

In this section we treat three compositions defined over functions with a unit delay, i.e. here each primitive function is assumed to have a unit time delay for its computation. These constructions are closely related each others (the difference lies in the treatment of constant functions).

Uniform composition

Kudryavcev initiated the theory of uniform composition [Kud60] The following lemma is proved in [Kud60], and explicit statement in this form is due to Nozaki [Noz78].

Table 5: Profiles of the classes under 2-line fixed coding.

	$n=1$	$n=2m>1$	$n=2m+1$
1	0	$2^{n+1}-2^{n/2+1}-2$	$2^{n+1}-3\cdot 2^{(n+1)/2}+2$
2	0	2	0
3	0	0	$2^{(n+1)/2}-3$
4,5	0	$2^{n/2}-2$	$2^{(n+1)/2}-3$
6	1	0	2
7,8	0	1	1
9	0	0	1
10,11	1	1	1
12	1	0	0
sum	4	2^{n+1}	2^{n+1}

Table 6: Up-to profiles of the classes under 2-line fixed coding.

	$n>1$
1	$2^{n+2}-2^2(2^{\lfloor n/2\rfloor}+3\cdot 2^{\lfloor (n-1)/2\rfloor})+7-(-1)^n$
2	$2\lfloor n/2\rfloor$
3	$2^{\lfloor (n+3)/2\rfloor}-3\lfloor (n-1)/2\rfloor-4$
4,5	$(3+(1+(-1)^n)/2)2^{\lfloor (n+1)/2\rfloor}-2n-\lfloor (n-1)/2\rfloor-4$
6	$2\lfloor (n-1)/2\rfloor+1$
7,8	$n-1$
9	$\lfloor (n-1)/2\rfloor$
10,11	n
12	1
sum	$2^{n+2}-4$

Table 7: Number of s-bases consisting of n-ary functions under 2-line fixed coding.

	n even	n odd
N^n	$3\cdot 2^n+2\cdot 2^{n/2}-9$	$2^{n+3}-9\cdot 2^{(n+1)/2}+5$
N_1^n	$2^{n+1}-2\cdot 2^{n/2}-2$	$2^{n+1}-3\cdot 2^{(n+1)/2}+2$
N_2^n	$2^n+4\cdot 2^{n/2}-7$	$3\cdot 2^{n+1}+6\cdot 2^{(n+1)/2}-4$
N_3^n	0	7

Table 8: Classes of symmetric functions under uniform compositions.

	ord. class	T_0T_1SLM	$M'XK$
1	#1	1 1 1 1 1	1 0 1
2	#1	1 1 1 1 1	0 0 1
3	#2	1 1 0 1 1	1 0 1
4	#2	1 1 0 1 1	0 0 1
5	#3	0 1 1 1 1	1 1 0
6	#4	1 0 1 1 1	1 1 0
7	#5	1 1 0 0 1	1 0 1
8	#5	1 1 0 0 1	0 0 1
9	#6	1 0 1 0 1	1 1 0
10	#7	0 1 1 0 1	1 1 0
11	#8	0 0 1 1 1	1 1 1
12	#9	1 0 1 0 0	0 1 0
13	#10	0 1 1 0 0	0 1 0
14	#11	0 0 1 1 0	1 1 1
15	#12	0 0 0 1 1	1 1 1
16	#13	0 0 0 1 0	1 1 1
17	#14	0 0 0 0 1	1 1 1
18	#15	0 0 0 0 0	1 1 1

Lemma 4.1. [Kud60] *There are 8 maximal sets under uniform composition: T_0, T_1, S, L, M, M', X and K.*

Theorem 4.1. *There are exactly 118 classes of s-bases under uniform composition. The corresponding N^n is indicated in Table 11.*

Ibuki and Inagaki constructions

Ibuki and Inagaki constructions give 7 and 6 maximal sets which coincide with above sets except K and K, X, respectively. Although the classes coincide in all three cases, bases are different due to the extra coordinate. There are 93 and 82 classes of bases in Ibuki [Ibu68] and Inagaki [Ina82] cases, respectively. These observations are also valid for s-bases since there is a symmetric representative in each class. The corresponding formulas N^n for Ibuki and Inagaki constructions are indicated in Tables 12,13.

Table 9: Profiles of the classes under uniform composition.

class	$n=1$	n even	n odd >1
$1,11$	0	$2^{n-1}-n$	$2^{n-1}-2^{(n-1)/2}-n+1$
$2,14$	0	n	$n-1$
$3,15$	0	0	$2^{(n-1)/2}-2$
$4,7,16,17$	0	0	1
$5,6$	0	$2^{n-1}-2$	$2^{n-1}-1$
$8,18$	1	0	0
$9,10$	0	1	0
$12,13$	1	1	1
sum	4	2^{n+1}	2^{n+1}

Table 10: Up-to profiles of the classes under uniform delay compositions.

class	Number of at most n-ary symmetric functions
1, 11	$2^n - 2^{\lfloor (n+1)/2 \rfloor} - \lfloor n^2/2 \rfloor$
2, 14	$\lfloor n^2/2 \rfloor$
3,15	$2^{\lfloor (n+1)/2 \rfloor} - 2\lfloor (n-1)/2 \rfloor - 2$
4,7,16,17	$\lfloor (n-1)/2 \rfloor$
5, 6	$2^n - n - 1 - \lfloor n/2 \rfloor$
8,18	1
9,10	$\lfloor n/2 \rfloor$
12,13	n
sum	$2^{n+2} - 4$

Table 11: Number of s-bases consisting of n-ary functions under uniform composition.

	n even	n odd
N^n	$2^{3(n-1)} + 3 \cdot 2^{2(n-1)} - 3n$	$2^{3(n-1)} + 3 \cdot 2^{2(n-1)} - 2^{n-1}$ $+4 \cdot 2^{(n-1)/2} - 5$
N_1^n	0	0
N_2^n	$3 \cdot 2^{2(n-1)} - 2n$	$3 \cdot 2^{2(n-1)} - 2^{n-1} - 2n - 2$
N_3^n	$2^{3(n-1)} - n$	$2^{3(n-1)} + 4 \cdot 2^{(n-1)/2} + n - 3$
N_4^n	0	n

Table 12: Number of s-bases consisting of n-ary functions (Ibuki construction).

	n even	n odd
N^n	$2^{2n} + 2^{n+1} - 3n - 4$	$2^{2n} + 2 \cdot 2^{(n+1)/2} - 6$
N_1^n	0	0
N_2^n	$2^{2n} - 2^n - 4$	$2^{2n} - 2^{n-1} - 2n - 3$
N_3^n	$2^{n+1} - n$	$2^{n-1} + 2 \cdot 2^{(n+1)/2} + n - 3$
N_4^n	0	n

Table 13: Number of s-bases consisting of n-ary functions (Inagaki construction).

	n even	n odd
N^n	$2^{2n-2} + (3n+5)2^{n-1}$ $-4n - 4$	$2^{2n-2} + 3 \cdot 2^{3(n-1)/2} + (3n-2)2^{n-1}$ $+(n+2)2^{(n-1)/2} - 4n - 2$
N_1^n	$2^{n-1} - n$	$2^{n-1} - 2^{(n-1)/2} - n + 1$
N_2^n	$2^{2n-2} + 3n \cdot 2^{n-1}$ $-2n - 4$	$2^{2n-2} + 3 \cdot 2^{3(n-1)/2} + (3n-2)2^{n-1}$ $+(n+2)2^{(n-1)/2} - 4n - 2$
N_3^n	$2^{n+1} - n$	$2^{n-1} - 2 \cdot 2^{(n+1)/2} - n - 1$
N_4^n	0	n

Table 14: Classes of symmetric functions under s-completeness.

	N_0N_1SLMM'	symmetric functions
1.	1 1 1 1 1 1	the remaining symmetric functions.
2.	1 1 1 1 1 0	omitted.
3.	1 1 1 0 1 1	$a + x_1 + \ldots + x_n$ $(n = 2m \geq 2), a \in \{0, 1\}$.
4.	1 1 0 1 1 1	omitted.
5.	1 0 1 1 1 1	$s^n_R, 2r < n$ and $s^n_R \notin M$ (omitted).
6.	0 1 1 1 1 1	$s^n_R, 2r_1 > n$ and $s^n_R \notin M$ (omitted).
7.	1 1 0 1 1 0	$s^n_{0,1,\ldots,(n-1)/2}$: n odd.
8.	1 1 0 0 1 1	$a + x_1 + \ldots + x_n (n = 2m + 1 \geq 3), a \in \{0, 1\}$.
9.	1 0 1 1 0 1	omitted.
10.	0 1 1 1 0 1	omitted.
11.	1 1 0 0 1 0	$1 + x$.
12.	1 0 1 0 0 0	c^n_1.
13.	0 1 1 0 0 0	c^n_0.
14.	0 0 0 1 0 1	$s^n_{(n+1)/2,\ldots,n}$: n odd.
15.	0 0 0 0 0 1	x.
16.	1 1 1 1 0 1	ϕ

5. S-bases under sequential circuit composition

Compositions allowing loops by using unit delay primitives and the notion of s-completeness are introduced by Nozaki [Noz84] (s- for sequential circuit).

Lemma 5.1. [Noz84] *There are exactly the following 6 maximal sets under s-completeness: N_0, N_1, S, L, M and M'.*

The classification of P_2-functions in this case was given in [MIS85]. There are exactly 16 P_2-classes, however, in class $\overline{N_0}\overline{N_1}\overline{SLM}\overline{M'}$ (16th) there is no symmetric function (for example, non-symmetric function $x_1x_2 \vee x_3x_4$ belongs to this class and there is no such example for $n < 4$).

Lemma 5.2. *There is no symmetric representative in the above class 16.*

This is the only case we have observed so far that the classes of symmetric functions and those of all functions of P_2 do not coincide (however, in P_3 there exist no symmetric function in 12 among 406 classes under ordinary composition [Sto87]).

Theorem 5.1. *There are 15 classes of symmetric functions under s-completeness (Table 14) and their profiles are given in Tables 15,16.*

Theorem 5.2. *There are exactly 50 classes of s-bases under s-completeness. The corresponding formula for N^n is indicated in Table 17.*

Table 15: Profiles of the classes under s-completeness.

class	$n=1$	n even	n odd >1
1	0	$2^{n+1}-2^{n/2+1}-n-2$	$2^{n+1}-3\cdot 2^{(n+1)/2}-n+3$
2	0	n	$n-1$
3	0	2	0
4	0	0	$2^{(n+1)/2}-4$
5,6	0	$2^{n/2}-n/2-1$	$2^{(n+1)/2}-(n+1)/2-1$
7,14	0	0	1
8	0	0	2
9, 10	0	$n/2$	$(n-1)/2$
11,15	1	0	0
12,13	1	1	1
sum	4	2^{n+1}	

Table 16: Up-to profiles of the classes under s-completeness.

class	$n\geq 1$
1	$2^{n+2}-(9+(-1)^n)2^{\lfloor (n+1)/2\rfloor}-\lfloor n^2/2\rfloor-4\lfloor n/2\rfloor+2n+6$
2	$\lfloor n^2/2\rfloor$
3	$2\lfloor n/2\rfloor$
4	$2^{\lfloor (n-1)/2\rfloor+2}-4\lfloor (n-1)/2\rfloor-4$
5,6	$(7+(-1)^n)2^{\lfloor (n-1)/2\rfloor}-(1/2)\lfloor n^2/2\rfloor-\lfloor (n-1)/2\rfloor-n-5$
7,14	$\lfloor (n-1)/2\rfloor$
8	$2\lfloor (n-1)/2\rfloor$
9, 10	$\lfloor n^2/2\rfloor/2$
11,15	1
12,13	n
sum	$2^{n+2}-4$

Table 17: Number of s-bases consisting of n-ary functions under s-completenss.

	n even	n odd
N^n	$3\cdot 2^n+(n+1)2^{n/2+1}$ $-n^2/4-2n-7$	$3\cdot 2^{n+1}+(7n-9)2^{(n-1)/2}$ $-(n^2+34n-3)/4$
N_1^n	$2^{n+1}-2^{n/2+1}-n-2$	$2^{n+1}-3\cdot 2^{(n+1)/2}-n+3$
N_2^n	$2^n+(n+2)2^{n/2+1}-n^2/4-n-5$	$2^{n+2}+(7n-3)2^{(n-1)/2}$ $-(n^2+30n+49)/4$
N_3^n	0	10

Table 18: Numbers of s-bases consisting solely of n-ary symmetric functions under several construction.

n	2	3	4	5	6	7	8	9	10
ordinary composition	2	36	72	446	1,078	5,634	16,628	77,834	263,154
r-line	7	15	31	63	127	255	511	1,023	2,047
2-line fix	7	33	47	189	199	885	791	3,813	3,127
uniform composition	14	111	692	4,859	35,822	274,395	2,146,280	16,973,627	135,004,130
Ibuki composition	14	66	272	1,034	4,202	16,410	66,020	262,202	1,050,590
Inagaki composition	14	64	180	662	1,732	6,890	20,060	84,362	280,020
sequential	12	45	69	248	276	1,017	1,017	3,840	3,724

6. Concluding remarks

We have given the profile of each class for each of the known 7 constructions. By this we have given formulas for the number of s-bases consisting solely of n-ary functions. The numerical data for the small numbers of n are given in table 18. The rapid growth of uniform composition is mainly due to the existence of rank 3 s-bases each of which is consist of 3 classes each having $O(2^{n-1})$ profile. By the given data of up-to profiles of the classes we can calculate the formula $N^{\leq n}$ for the number of s-bases consisting of up to n-ary functions.

Classification and base consideration for another modification of algebra of logic ϕ^o proposed by Cejtlin [Cej69] was done in [Tos81]. Several other modifications of propositional algebras are considered in [Gin85]. The profiles of the functions (not symmetric functions) of the classes are not known except some of them [Krn65], because explicit formulas for the numbers of n-ary monotone or clique functions [PMN88] are not known. For many-valued cases, the problem is not yet considered except symmetric Sheffer functions for 3-valued case [Sto89].

7. Acknowledgments.

The authors are indebted to Professors Tadahiro Kitahashi, Akihiro Nozaki, Teruo Hikita and Hajime Machida for many helpful suggestions. They are also thankful to Mathematical Informatics Section of Electrotechnical Laboratory.

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