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A Difference Set Of A Cantor Set

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Abstract. An example of a regular Cantor set whose self-difference set is a Cantor set with a positive measure is given. This is a counter example of one of the questions related to the homoclinic bifurcation of surface diffeomorphisms.

§.0 Introduction.

In [2], Palis–Takens investigated the homoclinic bifurcations of surface diffeomorphisms in the following context. Let M be a closed 2-dimensional manifold. We say a C^r-diffeomorphism $\phi : M \to M$ is *persistently hyperbolic* if there is a C^r-neighborhood $\mathcal{U}$ of ϕ and for every $\psi \in \mathcal{U}$, the non-wandering set $\Omega(\psi)$ is a hyperbolic set (refer [1] for the definitions and the notations of the terminologies of dynamical systems). Let $\{\phi_\mu\}_{\mu\in\mathbf{R}}$ be a 1-parameter family of C^2-diffeomorphisms on M. We define $\{\phi_\mu\}_{\mu\in\mathbf{R}}$ has a *homoclinic Ω-explosion* at $\mu = 0$ if:

i) For $\mu < 0$, ϕ_μ is persistently hyperbolic;

ii) For $\mu = 0$, the non-wandering set $\Omega(\phi_0)$ consists of a (closed) hyperbolic set $\widetilde{\Omega}(\phi_0) = \lim_{\mu\uparrow 0}\Omega(\phi_\mu)$ together with a homoclinic orbit of tangency $\mathcal{O}$ associated with a fixed saddle point p, so that $\Omega(\phi_0) = \widetilde{\Omega}(\phi_0) \cup \mathcal{O}$; the product of the eigenvalues of $d\phi_0$ at p is different from one in norm;

iii) The separatrices have quadratic tangency along $\mathcal{O}$ unfolding generically; $\mathcal{O}$ is the only orbit of tangency between stable and unstable separatrices of periodic orbits of ϕ_0.

Let Λ be a basic set of a diffeomorphism on M. $d^s(\Lambda)$ ($d^u(\Lambda)$) denotes the Hausdorff dimension in the transversal direction of the stable (unstable) foliation of stable (unstable) manifold of Λ (refer [2] for the precise definition), and is called the stable (unstable) *limit capacity*. B denotes the set of values $\mu > 0$ for which ϕ_μ is not persistently hyperbolic.

The result of Palis–Takens is;

THEOREM [2]. *Let $\{\phi_\mu;\ \mu \in \mathbf{R}\}$ be a family of diffeomorphisms of M with a homoclinic Ω-explosion at $\mu = 0$. Suppose that $d^s(\Lambda) + d^u(\Lambda) < 1$, where Λ is the basic set of ϕ_0 associated with the homoclinic tangency. Then*

$$\lim_{\delta \to 0} \frac{m(B \cap [0,\delta])}{\delta} = 0$$

where m denotes Lebesgue measure.

This result says that if $d^s(\Lambda)+d^u(\Lambda) < 1$, then the measure of the parameters for which bifurcation occurs is relatively small.

Now the case of $d^s(\Lambda) + d^u(\Lambda) > 1$ comes into question as the next step. In the proof of the theorem above, one of the essential points is a question of how two Cantor sets in the line intersect each other when the one is slid. In [3], Palis proposed the following questions.

(Q.1) For affine Cantor sets X and Y in the line, is it true that $X - Y$ either has measure zero or contains intervals ?

(Q.2) Same for regular Cantor sets.

For two subset X,Y of $\mathbf{R}$,

$$X - Y = \{\, x - y \mid x \in X,\ y \in Y \,\}.$$

This can also be written as;

$$X - Y = \{\, \mu \in \mathbf{R} \mid X \cap (\mu + Y) \neq \phi \,\},$$

namely $X - Y$ is the set of parameters for which X and Y have a intersection point when Y is slid on the line.

Cantor set Λ in $\mathbf{R}$ is called *affine, regular* or C^r for $1 \le r \le \infty$ if Λ is defined with finite number of expanding affine, C^2 or C^r maps respectively (see §2 Definition 5 for the rigorous definition).

Our result in this note is that there is a counter example of (Q.2), namely;

THEOREM. *There exists a C^∞-Cantor set Λ such that*

(i) $m(\Lambda - \Lambda) > 0$,

(ii) $\Lambda - \Lambda$ *is a Cantor set.*

In the succeeding sections , we give an outline of the proof of this theorem. The complete proof will appear elsewhere.

§.1 Definition of the Cantor sets $\Lambda(s)$, $\Gamma(s)$.

First of all, we define two cantor set depending on a sequence of real numbers.

DEFINITION 1. Let $I = [x_1, x_2]$ be a closed interval and λ a real number with $0 < \lambda < \frac{1}{2}$. We define,

$$I_0(\lambda; I) = [x_1, x_1 + \lambda(x_2 - x_1)]$$
$$I_1(\lambda; I) = [x_2 - \lambda(x_2 - x_1), x_2] \ .$$

DEFINITION 2 (CANTOR SET $\Lambda(s)$). Let $I^0 = [0,1]$ and $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ be a one sided sequence of real numbers with $0 < \lambda_i < \frac{1}{2}$ for all $i \geq 1$. We define the Cantor set $\Lambda(s)$ as follows.

Let $I_0^1 = I_0(\lambda_1; I^0)$, $I_1^1 = I_1(\lambda_1; I^0)$ and $I^1 = I_0^1 \cup I_1^1$. Δ_n denotes the set of all sequences of 0 and 1 of length n. When I_β^{n-1}'s are defined for all $\beta \in \Delta_{n-1}$, we define;

$$I_{\beta 0}^n = I_0(\lambda_n; I_\beta^{n-1})$$
$$I_{\beta 1}^n = I_1(\lambda_n; I_\beta^{n-1}) \ .$$

Inductively, we can define I_α^n for all $\alpha \in \Delta_n$ and for all $n \geq 0$. Define

$$I^n = \bigcup_{\alpha \in \Delta_n} I_\alpha^n$$

and

$$\Lambda(s) = \bigcap_{n \geq 0} I^n \ .$$

This is clearly a Cantor set by the definition.

Next, we define another Cantor set $\Gamma(s)$.

Definition 3. Let $J = [x_1, x_2]$ and $0 < \lambda < \frac{1}{3}$. We define,

$$J_0(\lambda; J) = [x_1, x_1 + \lambda(x_2 - x_1)]$$
$$J_1(\lambda; J) = [\frac{x_1 + x_2}{2} - \frac{\lambda}{2}(x_2 - x_1), \frac{x_1 + x_2}{2} + \frac{\lambda}{2}(x_2 - x_1)]$$
$$J_2(\lambda; J) = [x_2 - \lambda(x_2 - x_1), x_2] \ .$$

Definition 4. Let $J^0 = [-1, 1]$ and $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ be a one sided sequence of real numbers with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$. Let

$$J_0^1 = J_0(\lambda_1; J^0)$$
$$J_1^1 = J_1(\lambda_1; J^0)$$
$$J_2^1 = J_2(\lambda_1; J^0)$$

and Π_n denote the set of all sequences of $0, 1, 2$ of length n. When J_δ^{n-1}'s are defined for all $\delta \in \Pi_{n-1}$, we define;

$$J_{\delta 0}^n = J_0(\lambda_n; J_\delta^{n-1})$$
$$J_{\delta 1}^n = J_1(\lambda_n; J_\delta^{n-1})$$
$$J_{\delta 2}^n = J_2(\lambda_n; J_\delta^{n-1}) \ .$$

Inductively, we can define J_γ^n for all $\gamma \in \Pi_n$ and for all $n \geq 0$. Define

$$J^n = \bigcup_{\gamma \in \Pi_n} J_\gamma^n$$

and

$$\Gamma(s) = \bigcap_{n \geq 0} J^n \ .$$

These cantor sets have the following relation.

Theorem 1. *Let $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ be a sequence of real numbers with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$. Then,*

$$\Lambda(s) - \Lambda(s) = \Gamma(s) \ .$$

§.2 Outline of the proof.

DEFINITION 5. Let Λ be a Cantor set on a closed interval I. Λ is called *affine, regular* or *C^r-Cantor set* for $1 \leq r \leq \infty$ if there are closed disjoint intervals $I_1, \cdots, I_k$ on I and onto affine, C^2 or C^r-maps $f_i : I_i \to I$ for all $1 \leq i \leq k$ such that;

(i) $|f'_i(x)| > 1 \quad \forall x \in I_i$

(ii) $\Lambda = \bigcap_{n=0}^{\infty} \{ \bigcup_{\sigma \in \Sigma_n^k} f_{\sigma(1)}^{-1} f_{\sigma(2)}^{-1} \cdots f_{\sigma(n)}^{-1}(I) \}$,
where $\Sigma_n^k = \{\sigma : \{1, \cdots, n\} \to \{1, \cdots, k\}\}$.

Our main result is restated as follows.

THEOREM 2. *There exists a sequence of real numbers $s = (\lambda_1, \lambda_2, \lambda_3, \cdots)$ with $0 < \lambda_i < \frac{1}{3}$ for all $i \geq 1$ such that;*

(i) $\Lambda(s)$ is a C^∞-Cantor set,

(ii) $m(\Lambda(s) - \Lambda(s)) > 0$,
where $m(\ )$ denotes the Lebesgue measure.

From now on, we shall give the outline of the proof of this Theorem 2.

Let $\{r_n\}_{n \geq 0}$ be a sequence of positive real numbers such that

(1) $$\sum_{n=0}^{\infty} r_n < 1 .$$

We define $\{\lambda_n\}_{n \geq 1}$ using this $\{r_n\}_{n \geq 0}$ as follows.

(2) $$\begin{cases} \lambda_1 = \dfrac{1}{3}(1 - r_0) \\ \lambda_{n+1} = \dfrac{1}{3}\left(\dfrac{1 - \sum_{i=0}^{n} r_i}{1 - \sum_{i=0}^{n-1} r_i}\right) \end{cases}$$

It is easily seen that

(3) $$0 < \lambda_n < \frac{1}{3} \qquad \forall n \geq 1 .$$

These numbers has the following relations.

LEMMA 3.
$$\sum_{i=0}^{n} r_i = 1 - 3^{n+1} \prod_{j=1}^{n+1} \lambda_j \qquad \forall n \geq 0 \ .$$

LEMMA 4.
$$r_n = 3^n(1 - 3\lambda_{n+1}) \prod_{j=1}^{n} \lambda_j \qquad \forall n \geq 0 \ .$$

where, we assume $\prod_{j=1}^{0} \lambda_j = 1$ for the simplicity of notation.

Using these lemmas, we can show (ii) of Theorem 2. In fact, the following lemma holds.

LEMMA 5. *Let $\{r_n\}_{n\geq 0}$ be a sequence of positive real numbers such that $\sum_{n=0}^{\infty} r_n < 1$, and $\{\lambda_n\}_{n\geq 1}$ be the sequence defined by (2). Then, $m(\Gamma(s)) > 0$.*

§.3 The regularity of $\Lambda(s)$.

We define a sequence $\{r_n\}_n \geq 0$ (and so $\{\lambda_n\}_{n\geq 1}$), and prove that $\Lambda(s)$ is C^∞. First of all, we fix a C^∞–function $h(t)$ on $[0,1]$ with the following properties.

(i) $h(t) \geq 0$,

(ii) $\int_0^1 h(t)dt = 1$,

(iii) for all $n \geq 0$,
$$\begin{cases} \lim\limits_{t\downarrow 0} h^{(n)}(t) = 0 \ , \\ \lim\limits_{t\uparrow 1} h^{(n)}(t) = 0 \ . \end{cases}$$

To define $\{r_n\}_{n\geq 0}$, we define the following sequences. For each integers $n \geq 0$, let
$$q_n = \max\{q_0, q_1, \cdots, q_{n-1}, 1, \sup_{t\in[0,1]} |h^{(n)}(t)| \} \ .$$

For $n \geq 0$, we define,
$$r_n = \frac{4^{-(n^2+2)}}{q_n}$$

Since $r_n \leq 4^{-(n^2+2)} \leq 4^{-(n+2)}$, we have,

$$\sum_{n=0}^{\infty} r_n < \sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{12} . \tag{4}$$

Therefore, $\{r_n\}_{n\geq 0}$ satisfy (1). We define another sequence of positive real numbers;

$$m_n = \frac{3(3r_{n-1} - r_n)}{2^{n-1}(1 - \sum_{i=0}^{n-1} r_i)} \qquad \forall n \geq 1 .$$

Since $\{r_n\}_{n\geq 0}$ is monotonically decreasing and by (4), $m_n > 0$ for all $n \geq 1$.

U^0 denotes the open interval between I_0^1 and I_1^1, namely;

$$U^0 = I^0 \backslash (I_0^1 \cup I_1^1) .$$

In general, U_α^{n-1} ($\alpha \in \Delta_{n-1}$) denotes the open interval between $I_{\alpha 0}^n$ and $I_{\alpha 1}^n$ in I_α^{n-1}, namely;

$$U_\alpha^{n-1} = I_\alpha^{n-1} \backslash (I_{\alpha 0}^n \cup I_{\alpha 1}^n) .$$

Let $\ell_n = \ell(I_\alpha^n)$. Then, by the definition,

$$\ell_n = \lambda_n \ell_{n-1} .$$

Let $u_n = \ell(U_\alpha^n)$, and $U_\alpha^n = [x_\alpha, y_\alpha]$. Then,

$$u_n = \ell_n - 2\ell_{n+1} ,$$

and

$$u_n = y_\alpha - x_\alpha .$$

We prove the smoothness of $\Lambda(s)$ as follows. We define a non-negative C^∞–function $f(t)$ on $[0, \lambda_1]$ and define;

$$g(t) = \int_0^t (f(s) + 3) ds .$$

We put;

$$\begin{cases} g_0(t) = g(t) & on\ [0, \lambda_1] \\ g_1(t) = g(t - 1 + \lambda_1) & on\ [1 - \lambda_1, 1] . \end{cases}$$

and prove that these g_0 and g_1 define $\Lambda(s)$.

DEFINITION OF $f(t)$. Recall that we have already defined a C^∞–function $h(t)$ on $[0,1]$. We define $f(t)$ using this $h(t)$ as follows. Let $[x'_\alpha, y'_\alpha]$ be the interval of length $\frac{\ell_n}{3}$ in the middle of U^n_α such that

$$[x'_\alpha, y'_\alpha] = [x_\alpha + \frac{1}{2}(u_n - \frac{\ell_n}{3}), y_\alpha - \frac{1}{2}(u_n - \frac{\ell_n}{3})] .$$

We define $f(t)$ as follows.

(i) On U^n_α $(n \neq 0)$,

$$\begin{cases} f(t) = m_n h(\frac{t - x'_\alpha}{\frac{\ell_n}{3}}) & t \in [x'_\alpha, y'_\alpha] \\ f(t) = 0 & otherwise . \end{cases}$$

(ii) On $\Lambda(s)$, $f(t) = 0$.

What we have to show are;

(I) $f(t)$ is a C^∞–function on $[0, \lambda_1]$.

(II) g_0 and g_1 define $\Lambda(s)$.

To show the smoothness of $f(t)$, we define a function $f_n(t)$ for any $n \geq 0$ as follows. Since $f(t)$ is C^∞ on $U = \cup_{n\geq 1, \alpha\in\Delta_n} U^n_\alpha$ ($= [0,\lambda_1]\backslash\Lambda(s)$), $f^{(n)}(t)$ exists for all $n \geq 0$ on U. We define,

$$\begin{cases} f_n(t) = f^{(n)}(t) & for \quad t \in U \\ f_n(t) = 0 & otherwise \text{ (} i.e. \quad t \in \Lambda(s) \text{).} \end{cases}$$

The smoothness is shown by proving that;

LEMMA 6. *For any $n \geq 0$, f_n is differentiable at any $t \in [0, \lambda_1]$ and $f'_n(t) = f_{n+1}(t)$.*

For the proof of (II), we need some lemmas. Let $I^n_\alpha = [r^n_\alpha, s^n_\alpha]$.

LEMMA 7. *For all $\alpha, \alpha' \in \Delta_n$,*

$$\int_{I^n_\alpha} f(t)dt = \int_{I^n_{\alpha'}} f(t)dt .$$

LEMMA 8. *For all* $n \geq 1$,

$$\int_0^{\ell_n} f(t)dt = \frac{1}{3}m_n\ell_n + 2\int_0^{\ell_{n+1}} f(t)dt \ .$$

LEMMA 9. *For all* $n \geq 1$,

$$\ell_{n-1} = g_0(\ell_n) \ .$$

We have to prove that,

$$\Lambda(s) = \bigcap_{n\geq 0}\{ \bigcup_{\sigma\in\Sigma_n^2} g_{\sigma(1)}^{-1}g_{\sigma(2)}^{-1}\cdots g_{\sigma(n)}^{-1}(I^0)\} \ .$$

Recall that $\Sigma_n^2 = \{0,1\}^{\{1,\cdots,n\}}$ and $I^0 = [0,1]$. This is obtained by showing the following lemma.

LEMMA 10. *For all* $n \geq 0$ *and* $\alpha \in \Delta_n$,

$$g_0(I_{0\alpha}^{n+1}) = I_\alpha^n \ , \qquad g_1(I_{1\alpha}^{n+1}) = I_\alpha^n \ .$$

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