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A COUNTER EXAMPLE OF STRONG BAUM-CONNES CONJECTURES FOR FOLIATED MANIFOLDS

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§1 Introduction Related to a new index theory, Baum and Connes conjectured in [1] that the analytic and topological K-theory for foliations and dynamical systems are isomorphic each other under the K-theoretic index map. Since then, there appeared several papers supporting the conjecture (cf:[1]~[9] except [4]). However, Skandalis recently showed in [4] that there exists a counter example of the strong Connes-Kasparov conjecture for K-theory of C^*-crossed products. In his proof, a central tool is the property T due to Kazhdan in semisimple Lie groups of real rank one.

Modifying his idea, we shall show in this note that there exists a counter example of the strong Baum-Connes conjecture for foliated manifolds.

§2 Preliminaries Let (M,F) be a foliated manifold and G its holonomy groupoid. Taking the source and range maps s,r from G to M respectively, one can define the foliation $\tilde{F}$ of G coming from the tensor product of the pull backs $s^*(F)$ and $r^*(F)$ of F by s and r respectively. Let $\Omega^{1/2}$ be the half density bundle over G tangential to $\tilde{F}$ and denote by $C_c(\Omega^{1/2})$ the *-algebra of all continuous sections of $\Omega^{1/2}$ with compact support. The *-algebraic operation is defined as follows:

$$(f\cdot g)(\gamma) = \int_{\gamma=\gamma_1\gamma_2} f(\gamma_1)g(\gamma_2)$$

$$f^*(\gamma) = \bar{f}(\gamma^{-1})$$

for all $f,g \in C_c(\Omega^{1/2})$ where $\bar{f}(\gamma) = \overline{f(\gamma)}$. Given any $x \in M$, let H_x be the Hilbert space consisting of all L^2-sections of $\Omega^{1/2}$ over G and π_x the *-representation of $C_c(\Omega^{1/2})$ on H_x defined by

$$(\pi_x(f)\xi)(\gamma) = \int_{\gamma=\gamma_1\gamma_2} f(\gamma_1)\xi(\gamma_2)$$

for all $f \in C_c(\Omega^{1/2})$ and $\xi \in H_x$. The completion $C_r^*(M,F)$ of $C_c(\Omega^{1/2})$ with respect to $\|f\| = \sup_{x\in M} \|\pi_x(f)\|$ is called a foliation C^*-algebra associated to (M,F).

We now consider the K-theory $K_a(M,F)$ of $C_r^*(M,F)$ which is called the <u>analytic</u> K-theory of (M,F). We also have another K-theory of (M,F) using a pure topological way. It is called the <u>topological</u> K-theory of (M,F) which is denoted by $K_t(M,F)$. By Baum-Connes[1], if G is torsion free, then $K_t(M,F)$ is nothing more than the twisted K-theory $K^{\nu}(BG)$ of the classifying space BG of G by the transverse bundle ν of F. More precisely, the latter is the K-theory $K(B\tilde{\nu}/S\tilde{\nu})$ of the Thom space $B\tilde{\nu}/S\tilde{\nu}$ of the ν-bundle $\tilde{\nu}$ over BG. We then state the Baum-Connes conjecture for foliated manifolds as follows:

<u>Baum-Connes Conjecture:</u> Given a smooth foliated manifold (M,F), the K-index map is an isomorphism from $K_t(M,F)$ to $K_a(M,F)$.

There exists several papers supporting the above conjecture (cf:[1]~[9] except [4]).

Suppose G is torsion free and ν has a spin^c structure, then we know that $K^{\nu}(BG)$ is equal to K(BG) via Thom isomorphism. Related to the conjecture, we may formulate the following conjecture which is a strong version of the Baum-Connes' one:

Strong Baum-Connes Conjecture: Given any smooth foliated manifold (M,F) where G is torsion free and ν is spin^c, then the K-index map is an KK-isomorphism from $K_t(M,F)$ to $K_a(M,F)$ in the sense of Kasparov.

§3 Construction Let G be a connected semisimple linear Lie group which is locally isomorphic to Sp(n,1) or F_4 $(n \geq 2)$. Then it is of real rank one without Kazhdan's property T. Thanks to Skandalis[4], it is non K-nuclear, which means that the K-theory of the reduced group C^*-algebra is no longer KK-isomorphic to that of nuclear C^*-algebras. Using this fact, he found a counter example of the strong Connes-Kasparov conjecture for C^*-crossed products. We now modify his idea in order to find a counter example of the strong Baum-Connes conjecture for foliated manifolds.

Let V be a vector group on which G acts faithfully. By Borel's result [12], there exist torsion free uniform lattices Γ, Δ of G, V respectively such that Δ is Γ-invariant. By Wang's result [10], the semidirect product $V \times_s G$ of V by G has the property T which is no longer of real rank one. Since $\Delta \times_s \Gamma$ is a uniform lattice of $V \times_s G$, it also has the property T. By the similar way as in the proof of Skandalis [4], we have the following lemma which is of independent interest:

Lemma 1. $C_r^*(\Delta \times_s \Gamma)$ is non KK-nuclear.

Remark Dr.Matsumoto generalized the above lemma in more general setting.

Since Δ is a torsion free uniform lattice of V, the character group $\hat{\Delta}$ of Δ is a torus which may be chosen as of even dimension.

Let H be a maximal compact subgroup of G and $M = (G/H)\times_\Gamma\hat{\Delta}$ the orbit space of $(G/H)\times\hat{\Delta}$ by the diagonal Γ-action. As Γ is torsion free, M is a smooth manifold. Let $F = \{ (G/H)\times_\Gamma\{t\} \mid t \in \hat{\Delta} \}$. Then we have the following lemma:

Lemma 2. (M,F) is a foliated $\hat{\Delta}$ - bundle over $\Gamma\backslash G/H$.

Let us consider the reduced crossed product $(C(\hat{\Delta})\times_\alpha\Gamma)_r$ of $C(\hat{\Delta})$ by the holonomy action α of Γ on $\hat{\Delta}$ and $BC(\cdot)$ means the C^*-algebra of all compact operators on a Hilbert space $\cdot$. By the joint work with Natsume (cf:[11]), we have by Lemma 1 the following Lemma:

Lemma 3. $C_r^*(M,F)$ is isomorphic to $(C(\hat{\Delta})\times_\alpha\Gamma)_r \otimes BC(L^2(\Gamma\backslash G/H))$.

It then easily follows from Lemma 2 that

Corollary 4. $K_a(M,F) = K_a(\hat{\Delta},\Gamma) = K((C(\hat{\Delta})\times_\alpha\Gamma)_r)$.

In what follows, we shall determine $K_t(M,F)$. Let us consider the leaf ℓ_χ in F passing through $\chi \in \hat{\Delta}$. Then its fundamental group $\pi_1(\ell_\chi)$ is the stabilizer Γ_χ of Γ at χ. Since $\hat{\Delta}$ is the torus of even dimension, the holonomy map h_χ of ℓ_χ is a homomorphism from $\pi_1(\ell_\chi)$ into $\pi_1(\hat{\Delta})$. Therefore, $h_\chi(\Gamma_\chi)$ is torsion free for all $\chi \in \hat{\Delta}$, which means that G is also torsion free. By Baum-Connes [1], we then have the following lemma:

Lemma 5. $K_t(M,F) = K^{\nu}(BG)$ where $\nu = T(M)/F$.

In our case, as $T(M)/F = T(\hat{\Delta})$, ν has a complex structure. By Thom isomorphism, we deduce the following lemma:

Lemma 6. $K^{\nu}(BG) = K(BG)$.

Combining all the lemmas cited above, we obtain the following main theorem:

Theorem 7. Let G be a connected semisimple Lie group which is locally isomorphic to $Sp(n,1)$ or F_4 $(n \geq 2)$ acting on a vector group V faithfully. Let Γ, Δ be torsion free uniform lattices of G, V respectively such that Γ is Δ-invariant. If (M,F) is a foliated $\hat{\Delta}$-bundle over $\Gamma \backslash G/H$, then it is a counter example for the strong Baum-Connes conjecture where $\hat{\Delta}$ is the character group of Δ and H is a maximal compact subgroup of G.

Proof. By definition, the holonomy groupoid G of F has no torsion and $\nu = T(M)/F$ has a complex structure. Suppose the K-index map is a KK-isomorphism from $K_t(M,F)$ to $K_a(M,F)$. This means by Lemma 3,5 and 6 that there exists an invertible element of $KK((C(\hat{\Delta}) \times_{\alpha} \Gamma)_r, C_0(BG))$ which implement the K-index map where $C_0(BG)$ is the C^*-algebra of all continuous sections of $E\Gamma \times_{\Gamma} C(\hat{\Delta})$ vanishing at infinity. Since $C_0(BG)$ is nuclear and $(C(\hat{\Delta}) \times_{\alpha} \Gamma)_r = C^*_r(\Delta \times_s \Gamma)$ up to isomorphism, this contradicts to Lemma 1. Q.E.D.

Remark. In spite of the above theorem, the original Baum-Connes conjecture still remains open even for the above example.

References

[1] P.Baum-A.Connes, Geometric K-theory for Lie groups and foliations, Preprint (1982).

[2] A.Connes, A survey of foliations and operator algebras, Proc.Symp.Pure Math., 38(1982) Part 1, 521-628.

[3] T.Natsume, Topological K-theory for codimension 1 foliations without holonomy, Adv.Stud.Pure Math.,5 (1985), 15-27.

[4] G.Skandalis, Une notion de nuclearite en K-theorie, K-theory 1 (1988), 549-573.

[5] H.Takai, C*-algebras of Anosov foliations, Lec.Notes Math., Springer 1132 (1985), 509-516.

[6] _______, KK-thwory for the C*-algebras of Anosov foliations, Res.Notes Math.Ser.,Pitman 123 (1986), 387-399.

[7] _______, Baum-Connes conjectures and their applications, World Sci.Adv.Ser.Dyn.Sys., 5 (1987), 89-116.

[8] _______, On the Baum-Connes conjecture, Proc.fourth U.S.-Japan Seminar, (1988).

[9] A.M.Torpe, K-theory for the leaf space of foliations by Reeb components, Jour.Func.Anal., 61 (1985), 15-71.

[10] P.S.Wang, On isolated points in dual spaces of locally compact groups, Math.Ann., 218 (1975), 19-34.

[11] C.C.Moore and C.Schochet, Global analysis on foliation spaces, Math.Sci.Res.Inst.Publ.,Springer 9 (1988).

[12] A.Borel, Compact Clifford-Klein forms of symmetric spaces, Topology, 2 (1963), 111-122.