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ON KNESER TYPE THEOREM FOR FUNCTIONAL DIFFEREMTIAL EQUATIONS WITH THE PHASE SPACE $\mathscr{C}_\gamma$ IN BANACH SPACES

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§ 1. Introduction.

Let $R = (-\infty,\infty)$ and E be an infinite dimensional Banach space with norm $|\cdot|_E$. Let $X = E$ or R. Denote by $\mathscr{C}_\gamma^X$, $\gamma \in R$, the space of continuous functions $\psi : (-\infty,0] \to X$ having the limit $\lim_{\theta\to-\infty} e^{\gamma\theta}\psi(\theta)$ with the norm

$$|\psi|_{\mathscr{C}_\gamma^X} = \sup_{-\infty<\theta\le 0} e^{\gamma\theta}|\psi(\theta)|_X.$$

If $x : (-\infty,\sigma+a) \to X$, $0<a\le\infty$, then for any $t\in(-\infty,\sigma+a)$ we define $x_t : (-\infty,0] \to X$ by $x_t(\theta) = x(t+\theta)$, $-\infty<\theta\le 0$.

The purpose of this paper is to give Kneser type theorem on the set of solutions for the Cauchy problem of the functional differential equation(FDE) with infinite delay in a Banach space (for brevity, CP(1.1)),

$$\frac{dx}{dt} = f(t,x_t), \quad x_\sigma = \varphi \in \mathscr{C}_\gamma, \tag{1.1}$$

under the condition that $f : [\sigma,\sigma+a]\times\mathscr{C}^E_\gamma(\varphi,r) \to E$, $\mathscr{C}^E_\gamma(\varphi,r) := \{\psi\in\mathscr{C}^E_\gamma \mid |\varphi-\psi|_{\mathscr{C}^E_\gamma}\leq r\}$, is a uniformly continuous mapping. The argument in the proof of the main theorem(Theorem 3.4) is besed on the idea in [9] and on properties of $\mathscr{C}^E_\gamma$. Our result extends the one obtained in [10] and is closely related to the one due to Kubiaczyk [2] for ordinary differential equations(ODE's).

§ 2. Some Lemmas.

In this section, we shall show a differential inequality and a comparison theorem. For a continuous function $w : (a,b) \to R$ and for $t\in(a,b)$, $(D_+w)(t)$, $(\underline{D}_-w)(t)$ and $(\overline{D}_+w)(t)$ denote the right hand derivative, the left hand lower derivative and the right hand upper derivative, respectively.

<u>Lemma 2.1</u>. Let $w : [\sigma,\sigma+a) \to R$ be a continuous function such that $(D_+w)(t)$ exists for all $t\in[\sigma,\sigma+a)$. Then the following inequalities hold :

1)

$$\overline{D}_+ \sup_{\sigma\leq s\leq t} w(s) \leq |(D_+w)(t)|.$$

2) If $w(t) \geq 0$, then

$$\overline{D}_+ \sup_{\sigma\le s\le t} e^{\gamma(s-t)}w(s) \le \begin{cases} |(D_+w)(t)| & \text{if} \quad \gamma\ge 0 \\ |(D_+w)(t)| - \gamma \sup\limits_{\sigma\le s\le t} e^{\gamma(s-t)}w(s) & \text{if} \quad \gamma<0. \end{cases}$$

Proof. For a proof of the assertion 1) refer to [1,6]. Set $u(t) = \sup\{w(s)|\sigma\le s\le t\}$, $z(t) = \sup\{e^{\gamma s}w(s)|\sigma\le s\le t\}$ and $I=[\sigma,\sigma+a)$. Clearly, $z(t)$ is nondecreasing in $t\in I$. Let any $\tau\in I$ be a fixed number and $\gamma\in(-\infty,0)$. Then we have, for $h > 0$,

$$z(\tau+h) - z(\tau) = e^{\gamma t_0}w(t_0) - z(\tau) \quad \text{for some} \quad t_0\in[\tau,\tau+h]$$

$$\le e^{\gamma\tau} \sup_{\sigma\le s\le\tau+h} w(s) - e^{\gamma\tau} \sup_{\sigma\le s\le\tau} w(s)$$

$$= e^{\gamma\tau}\{u(\tau+h) - u(\tau)\},$$

from which it follows that

$$\overline{D}_+ \sup_{\sigma\le s\le t} e^{\gamma s}w(s) \le e^{\gamma t}\overline{D}_+ \sup_{\sigma\le s\le t} w(s).$$

It is easy to prove the assertion 2) in case where γ is a negative number. Let $\gamma \ge 0$. Then by the assertion 1) we have

$$\overline{D}_+ \sup_{\sigma\le s\le t} e^{\gamma(s-t)}w(s) \le -\gamma e^{-\gamma t} \sup_{\sigma\le s\le t} e^{\gamma s}w(s) + e^{-\gamma t}\overline{D}_+ \sup_{\sigma\le s\le t} e^{\gamma s}w(s)$$

$$\le -\gamma e^{-\gamma t} \sup_{\sigma\le s\le t} e^{\gamma s}w(s) + e^{-\gamma t}|D_+(e^{\gamma t}w(t))|$$

$$\leq |(D_+ w)(t)|$$

as required.

<u>Lemma 2.2</u>. Let $\gamma \geq 0$ and $U : [\sigma,\sigma+a]\times R^+ \to R^+$ be a continuous function, where $R^+ = [0,\infty)$. Assume that

(1) $u^* : [\sigma,\sigma+a] \to R^+$ is the maximal solution of the scalar differential equation

$$\frac{du}{dt} = U(t,u(t)), \quad u(\sigma) = u_0 \geq 0 \ ; \text{ and}$$

(2) $m : (-\infty,\sigma+a] \to R$ is a continuous function such that $m_\sigma \in \mathcal{C}_\gamma^R$ and $m(t) \geq 0$ on $[\sigma,\sigma+a]$, and that, for every $t_1 \in [\sigma,\sigma+a]$ such that $|m_{t_1}|_{\mathcal{C}_\gamma^R} = m(t_1)$, the differential inequality

$$(D_- m)(t_1) \leq U(t_1, m(t_1))$$

is satisfied.

If $|m_\sigma|_{\mathcal{C}_\gamma^R} \leq u_0$, then

$$m(t) \leq u^*(t) \quad \text{for} \quad t \in [\sigma,\sigma+a].$$

<u>Proof</u>. For any $\varepsilon > 0$ we denote by $u(t,\varepsilon)$ any solution of the differential equation

$$\frac{d}{dt}u(t) = U(t,u(t)) + \varepsilon, \quad u(\sigma) = u_0 + \varepsilon. \tag{2.1}$$

Then, by Lemma 1.3.1 in [4], we have

$$\lim_{\varepsilon\to 0+} u(t,\varepsilon) = u^*(t)$$

uniformly on $[\sigma,\sigma+a]$. Thus it is sufficient to show that for every $\varepsilon > 0$, sufficiently small,

$$m(t) \le u(t,\varepsilon) \quad \text{on} \quad [\sigma,\sigma+a].$$

Suppose, on the contrary, that the set

$$Z = \{t\in[\sigma,\sigma+a] \mid m(t) > u(t,\varepsilon)\}$$

is nonempty and define $t_1 = \inf Z$. Then we have $t_1 > \sigma$, because $|m_\sigma|_{\mathcal{C}^R_\gamma} \le u_0 < u_0+\varepsilon$. Moreover, since $m(t_1) = u(t_1,\varepsilon)$ and $m(t) < u(t,\varepsilon)$ for $t\in[\sigma,t_1)$, it is easy to see that

$$\underline{D}_-m(t_1) \ge \liminf_{h\to 0-} \frac{1}{h}\{u(t_1+h,\varepsilon) - u(t_1,\varepsilon)\}$$

$$= U(t_1,m(t_1)) + \varepsilon \quad \text{by (2.1)}.$$

Hence, we have

$$\underline{D}_-m(t_1) > U(t_1,m(t_1)). \tag{2.2}$$

On the other hand, since $U(t,s) \geq 0$ and $u(t,\varepsilon)$ is nondecreasing in t, we have

$$\begin{aligned}|m_{t_1}|_{\mathscr{C}^R_\gamma} &= \sup_{\theta\leq 0} e^{\gamma\theta}|m(t_1+\theta)| \\ &= \max\{ \sup_{\theta\leq\sigma-t_1} e^{\gamma\theta}|m(t_1+\theta)|, \sup_{\sigma-t_1\leq\theta\leq 0} e^{\gamma\theta}|m(t_1+\theta)|\} \\ &= \max\{\sup_{s\leq 0} e^{\gamma(s-t_1+\sigma)}|m(\sigma+s)|, m(t_1)\} \\ &= \max\{e^{\gamma(\sigma-t_1)}|m_\sigma|_{\mathscr{C}^R_\gamma}, m(t_1)\} \\ &= m(t_1).\end{aligned}$$

Thus, from the assumption 2) we are led to the inequality

$$\underline{D}_{-}m(t_1) \leq U(t_1,m(t_1)),$$

which is incompatible with (2.2). This implies that the set Z is empty. Therefore the proof is completed.

A function $\eta : (\sigma,\sigma+a]\times[0,2r] \to R$ is said to be a Kamke-type function if the following conditions hold :

(η_1) $\eta = \eta(t,s)$ is a real-valued function, defined on

$(\sigma,\sigma+a]\times[0,2r]$, which is Lebesgue measurable in t for each fixed $s\in[0,2r]$ and is continuous in s for *a.a.* $t\in[\sigma,\sigma+a]$:

(η_2) There exists a function α, defined on $(\sigma,\sigma+a]$ and locally integrable there, such that $|\eta(t,s)| \le \alpha(t)$ for *a.a.* $t\in(\sigma,\sigma+a]$ and all $s\in[0,2r]$.

The following result is a modification of the one given by [8, Lemma 3.1]. The proof is obvious.

Lemma 2.3. Let $\eta(t,s) : (\sigma,\sigma+a]\times[0,2r] \to R$ be a Kamke-type function and let $\{w^n\}$ and $\{z^n\}$ converge pontwise to functions w^0 and z^0 on $[\sigma,\sigma+a]$ as $n\to\infty$, respectively. Assume that

1) there are a constant $H > 0$ such that

$|w^n(t) - w^n(s)|\le H|t-s|$ for all $t,s\in[\sigma,\sigma+a]$ and all $n\in\mathbb{N}$; and

2) w^n and z^n are related to each other as

$$\frac{d}{dt} w^n(t) \le \eta(t,z^n(t)) + \sigma_n \qquad \text{for } a.a.\ t\in(\sigma,\sigma+a),$$

where $\sigma_n\ge 0$ and $\sigma_n \to 0$ as $n\to\infty$.

Then

$$\frac{d}{dt}w^0(t) \le \eta(t,z^0(t)) \qquad \text{for } a.a.\ t\in(\sigma,\sigma+a).$$

§ 3. Main results.

For a bounded set Ω of E, the α-measure of Ω is defined as follows :

$$\alpha(\Omega) = \inf\{d>0 \mid \Omega \text{ has a finite cover of diameter } < d\}.$$

Let $\mathfrak{X}$ be the set of functions x on $(-\infty,\sigma+a]$, $0<a<\infty$, into E such that x is continuous on $[\sigma,\sigma+a]$ and $x_\sigma \in \mathscr{C}_\gamma$. For a subset $\mathfrak{X} \subset \mathfrak{X}$, we will use the following notations :

$$\mathfrak{X}(t) = \{x(t)\in E \mid x\in\mathfrak{X}\}, \quad \mathfrak{X}_t = \{x_t \mid x\in\mathfrak{X}\} \quad \text{for } t\in[\sigma,\sigma+a)$$

and

$$\mathfrak{X}|[c,d] = \{x|[c,d] \mid x\in\mathfrak{X}\},$$

where $c,d\in[\sigma,\sigma+a)$ and $x|[c,d]$ is the restriction of x to $[c,d]$. We denote by $C([a,b],E)$ the set of all the continuous functions $x : [a,b] \to E$ with supremum norm. For brevity, we denote $\mathscr{C}_\gamma$ the phase space $\mathscr{C}_\gamma^E$ when $X = E$. The following lemma is concerned with the phase space $\mathscr{C}_\gamma$.

<u>Lemma 3.1</u> (Shin [7]). If $\mathfrak{X}_\sigma$ is relatively compact in $\mathscr{C}_\gamma$ and if $\mathfrak{X}|[\sigma,t]$ is a bounded and equicontinuous set in $C([\sigma,t],E)$, then

$$\alpha(\mathfrak{X}_t) = e^{-\gamma t} \sup_{\sigma \le s \le t} e^{\gamma s} \alpha(\mathfrak{X}(s)).$$

Lemma 3.2 (Shin [9]). Let $\{S_n\}$ be a family of nonempty bounded subsets of a Banach space Y such that $S_{n+1} \subset S_n$ for $n \in \mathbb{N}$. If S_n is connected for every $n \in \mathbb{N}$ and if $\alpha(S_n) \to 0$ as $n \to \infty$, then the set $\bigcap_{n=1}^{\infty} cl\, S_n$ is nonempty, compact and connected, where cl A stands for the closure of A.

Assume that $f : [\sigma, \sigma+a] \times \mathcal{B}_\gamma(\varphi, r) \to E$ is a uniformly continuous function such that $|f|_E \le M$. Then a function $u : (-\infty, \sigma+\xi] \to E$, $0 < \xi \le a$, said to be an $\frac{1}{n}$-approximate solution for CP(1.1) if the following conditions hold :

(1) u is continuous on J, $J = [\sigma, \sigma+\xi]$, and $u_\sigma = \varphi \in \mathcal{B}$;

(2) u has the right hand derivative $(D_+ u)(t)$ such that $|(D_+ u)(t)|_E \le M$ on $[\sigma, \sigma+\xi)$, and satisfies

$$u(t) = \varphi(0) + \int_\sigma^t (D_+ u)(s) ds \quad \text{for } t \in J \text{ ; and}$$

(3) $|(D_+ u)(t) - f(t, u_t)|_E \le \frac{1}{n}$ for $t \in [\sigma, \sigma+\xi)$.

We denote by $Q^n[d]$ the set of all the $\frac{1}{n}$-approximate solutions, defined on $(-\infty, \sigma+d]$, for CP(1.1). Then there is a $\xi > 0$ and the set $Q^n := Q^n[\xi]$ is nonempty(see [8, Lemma 2.1]).

Lemma 3.3 (Shin [9]). Let $f : [\sigma, \sigma+a] \times \mathcal{B}_\gamma(\varphi, r) \to E$ be unifomly continuous and $|f|_E \le M$ on $[\sigma,\sigma+a]\times\mathscr{C}_\gamma(\varphi,r)$. Then Q^n is nonempty and $Q^n|J$ is connected in $C(J,E)$ for every $n\in\mathbb{N}$.

Now, we state the main result in this paper, which is related to the result due to Kubiaczyk[2] for ODE's.

Theorem 3.4. Assume that $f : [\sigma,\sigma+a]\times\mathscr{C}_\gamma(\varphi,r) \to E$ is a uniformly continuous function such that $|f|_E \le M$ on $[\sigma,\sigma+a]\times\mathscr{C}_\gamma(\varphi,r)$, and that there exists a Kamke-type function $\omega(t,s) : (\sigma,\sigma+a]\times[0,2r] \to R^+$ such that

1) $\omega(t,s)$ is nondecreasing in s ;

2) $\omega(t,z(t)) \to 0$ as $t \to \sigma+0$ and $\int_\sigma^t \omega(s,z(s))ds < \infty$

whenever $z : [\sigma,\sigma+a] \to [0,2r]$ is an absolutely continuous function satisfying the condition $(D^+z)(\sigma) = z(\sigma) = 0$, where $(D^+z)(\sigma) := \lim_{t\to\sigma+} \frac{z(t)}{t-\sigma}$;

3) $z \equiv 0$ is the unique absolutely continuous function, mapping $[\sigma,\sigma+a]$ into R^+, which satisfies the initial condition $(D^+z)(\sigma) = z(\sigma) = 0$ and the scalar differential equation

$$\frac{dz}{dt} = \begin{cases} \omega(t,z(t)) & \gamma \ge 0 \\ \omega(t,z(t)) - \gamma z(t) & \gamma < 0 \end{cases}$$

for $a.a.$ $t\in(\sigma,\sigma+a)$; and

4)

$$\underline{D}_{-}\alpha(A(t)) := \liminf_{h \to 0-} \frac{1}{h}[\alpha(A(t)) - \alpha(\{x(t) - hf(t,x_t) : x \in A\})]$$

$$\leq \omega(t,\alpha(A_t))$$

for $a.a.$ $t \in (\sigma,\sigma+a]$ and for any subset $A \subset \mathfrak{X}$ such that $A|[\sigma,\sigma+a]$ is equicontinuous and that $A_t \subset \mathscr{C}_\gamma(\varphi,r)$ for all $t \in [\sigma,\sigma+a]$.

Then the set of all the solutions for CP(1.1) defined on J $(=[\sigma,\sigma+\xi])$ is nonempty, compact and connected in $C(J,E)$.

Proof. From Lemma 3.2 and Lemma 3.3 it is sufficient to see that $\alpha(Q^n|J) \to 0$ as $n \to \infty$. Since $Q^n|J$ is an equicontinuous subset of $C(J,E)$, we have $\alpha(Q^n|J) \leq \sup\{\alpha(Q^n_t) |\ t \in J\}$ by Theorem 2.1 in [5]. Thus we must prove that $\alpha(Q^n_t) \to 0$ uniformly on J as $n \to \infty$. From the properties of the α-measure of noncompactness, we have, for $t \in (\sigma,\sigma+\gamma]$ and $h > 0$,

$$\frac{1}{h}\{\alpha(Q^n(t)) - \alpha(Q^n(t-h))\}$$

$$\leq \frac{1}{h}\{\alpha(Q^n(t)) - \alpha(\{x(t) - hf(t,x_t) \mid x \in Q^n\})\}$$

$$+ \frac{1}{h}\,\alpha(\{x(t) - x(t-h) - hf(t,x_t) \mid x \in Q^n\}). \qquad (3.1)$$

By the uniform continuity of f, for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|f(s,\varphi_1) - f(t,\varphi_2)|_E \leq \varepsilon/2$ if $|t-s| < \delta$ and

$|\varphi_1-\varphi_2|<\delta$. Since $\{x_t^n \mid x^n \in Q^n\}$ is uniformly equicontinuous on $[\sigma,\sigma+\zeta]$, we have, for any $x \in Q^n$ and $h \in (0,\delta)$,

$$|x(t) - x(t-h) - hf(t,x_t)|$$

$$\le \left|\int_{t-h}^{t} [D_+x(s) - f(s,x_s)]ds\right| + \left|\int_{t-h}^{t} [f(s,x_s) - f(t,x_t)]ds\right|$$

$$\le \frac{h}{n} + \frac{\varepsilon}{2}h. \qquad (3.2)$$

Set $w^n(t) = \alpha(Q^n(t))$ and $z^n(t) = \alpha(Q_t^n))$ for all $t \in J$. Clearly, we have, for $t,s \in J$ and any $n \in \mathbb{N}$,

$$w^{n+1}(t) \le w^n(t), \quad |w^n(t) - w^n(s)| \le 2M|t-s|$$

and, Lemma 3.1,

$$z^n(t) = \sup_{\sigma \le s \le t} e^{\gamma(s-t)} w^n(s). \qquad (3.3)$$

Hence we get

$$z^{n+1}(t) \le z^n(t) \quad \text{and} \quad |z^n(t) - z^n(s)| \le 2M \sup_{-a\le\theta\le 0} e^{\gamma\theta}|t-s|.$$

These imply that the sequences $\{w^n(t)\}$ and $\{z^n(t)\}$ converges to functions $w^0(t)$ and $z^0(t)$ uniformly on J, respectively.

Now, we shall show that $z^0(t) \equiv 0$ on J. From (3.1),(3.2) and the assumption 4) it follows that

$$\frac{dw^n(t)}{dt} \le \omega(t,z^n(t)) + \frac{2}{n} + \varepsilon \qquad \text{for } a.a.\ t\in(\sigma,\sigma+\xi).$$

Using Lemma 2.3 and the relation (3.3), we can obtain

$$\frac{dw^0(t)}{dt} \le \omega(t,z^0(t)) + \varepsilon \quad \text{for } a.a.\ t\in(\sigma,\sigma+\xi) \tag{3.4}$$

and

$$z^0(t) = \sup_{\sigma\le s\le t} e^{\gamma(s-t)}w^0(s). \tag{3.5}$$

Moreover, it is easy to see that $(D_+z^0)(\sigma) = z^0(\sigma) = 0$. Using the assumption 2), we can put

$$u(t) = \int_\sigma^t \omega(s,z^0(s))ds + \varepsilon(t-\sigma),$$

from which it follows that $w^0(t) \le u(t)$ for $t\in J$. Therefore we can obtain

$$0 \le \frac{du}{dt} = \omega(t,z^0(t)) + \varepsilon \qquad \text{for } a.a.\ t\in(\sigma,\sigma+\xi).$$

Put $v(t) = \sup_{\sigma\le s\le t} e^{\gamma(s-t)}u(s)$. Then, by Lemma 2.1 we can see that

$$\frac{dv}{dt} \le \begin{cases} \omega(t,z^0(t)) + \varepsilon \\ \\ \omega(t,z^0(t)) - \gamma z^0(t) + \varepsilon \end{cases} \qquad \text{for } a.a.\ t\in(\sigma,\sigma+\xi), \qquad (3.6)$$

On the other hand, since $w^0(t) \le u(t)$, we have $z^0(t) \le v(t)$ by (3.5). Letting $\varepsilon \to 0$ and using the assumption 1), we see that the relation (3.6) becomes

$$\frac{dv}{dt} \le \begin{cases} \omega(t,v(t)) \\ \\ \omega(t,v(t)) - \gamma v(t) \end{cases} \qquad \text{for } a.a.\ t\in(\sigma,\sigma+\xi).$$

It is easy to see that $(D_+v)(\sigma) = v(\sigma) = 0$. Thus, by Lemma 4.1 in [6] and the assumption 3), we have $v(t) \equiv 0$ and so, $z^0(t) \equiv 0$. This implies $\alpha(Q^n|J) \to 0$ as $n \to \infty$. Hence the proof is complete.

Corollary 3.5. The conditions 1) - 4) in Theorem 3.4 can be replaced as follows :

1) the condition 3) in Theorem 3.4 is satisfied ;and

2)

$$\alpha(f(t,B)) \le \omega(t,\alpha(B)) \quad \text{for } a.a.\ t\in(\sigma,\sigma+a) \text{ and all } B\subset\mathscr{C}_\gamma(\varphi,r).$$

Combining the argument in the proof of Theorem 3.4 and Lemma 2.2, we have the following result.

Proposition 3.6. Let $\gamma \ge 0$. Assume that f :

$[\sigma,\sigma+a]\times\mathscr{C}_\gamma(\varphi,r) \to E$ is a uniformly continuous function such that $|f|_E \le M$ on $[\sigma,\sigma+a]\times\mathscr{C}_\gamma(\varphi,r)$, and that there exists a continuous function $\omega(t,s) : [\sigma,\sigma+a]\times[0,2r] \to R^+$ such that

1) for every $t_1\in[\sigma,\sigma+a]$ such that $\alpha(A_{t_1}) = \alpha(A(t_1))$, where A is as in Theorem 3.4, the differential inequality

$$\underline{D}_-\alpha(A(t_1)) \le \omega(t,\alpha(A(t_1)),$$

is satisfied ; and

2) $u(t) \equiv 0$ is the unique continuous function, mapping $[\sigma,\sigma+a]$ into $[0,2r]$, which satisfies the scalar differential equation

$$\frac{du(t)}{dt} = \omega(t,u(t)), \quad u(\sigma) = 0.$$

Then the conclusion of Theorem 3.4 remains valid.

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