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Context-free Languages in $X^+ \setminus Q$

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Abstract

Let X be a nonempty finite set, called an *alphabet*, such that $|X| > 1$. By Q we denote the set $\{f \in X^+ \mid f = g^i\ (i \geq 1,\ g \in X^+) \Rightarrow i = 1\}$. Moreover, for any $i \geq 1$, $Q^{(i)}$ means the set $\{f^i \mid f \in Q\}$. Let $L \subseteq X^*$. Then P_L is the following congruence relation on X^* :

$$u \equiv v\ (P_L) \Leftrightarrow {}^\forall x,y \in X^*\ (xuy \in L \Leftrightarrow xvy \in L)$$

If $u \equiv v\ (P_L)$ implies $u = v$, then $L \subseteq X^*$ is called a *disjunctive language*. Let $L \subseteq X^*$. If $X^*uX^* \cap L \neq \varnothing$ for any $u \in X^*$, then $L \subseteq X^*$ is called a *dense language*. It is easy to see that a disjunctive language is a dense language.

Now we collect some results and a problem related to context-free languages.

(C.1) There is an infinite context-free language $L \subseteq X^*$ such that $L \subseteq Q$.

(C.2) There is an infinite context-free language $L \subseteq X^*$ such that $L \subseteq Q^{(2)}$.

(C.3) Let $i \geq 3$. Is there an infinite context-free language $L \subseteq X^*$ such that $L \subseteq Q^{(i)}$?

(C.4) For any $n \geq 1$, there is an infinite context-free language $L \subseteq X^*$ such that $L \subseteq \bigcup_{1 \leq i \leq n} Q^{(i)}$ and $|L \cap Q^{(n)}| = \infty$.

Note that some of the above results do not hold for some classes of regular languages. For instance, there is no dense regular language $L \subseteq X^*$ such that $L \subseteq Q$ (compare with (DC.1)). We consider now the case of dense (or disjunctive) context-free languages.

(DC.1) There is a disjunctive context-free language $L \subseteq X^*$ such that $L \subseteq Q$.

(DC.2) Is there a dense (or disjunctive) context-free language $L \subseteq X^*$ such that $L \subseteq Q^{(2)}$?

(DC.3) Let $i \geq 3$. Is there a dense (or disjunctive) context-free language $L \subseteq X^*$ such that $L \subseteq Q^{(i)}$?

(DC.4) There is a disjunctive context-free language $L \subseteq X^*$ such that $L \subseteq Q \cup Q^{(2)}$, $|L \cap Q| = \infty$ and $|L \cap Q^{(2)}| = \infty$.

(DC.5) There is a disjunctive context-free language $L \subseteq X^*$ such that, for any $i \geq 1$, $|L \cap Q^{(i)}| = \infty$.

To solve the problems (C.3), (DC.2) and (DC.3), we will determine the structure of context-free languages in $X^+ \setminus Q$ and $Q^{(2)}$.

Theorem 1. *Let $L \subseteq X^*$ be a context-free language such that $L \subseteq X^+ \setminus Q$. Then $L_1 = L \cap Q^{(2)}$ is a context-free language and $L_2 = L \cap (\bigcup_{i \geq 3} Q^{(i)})$ is a regular language. More exactly, L_2 can be represented as follows :*

$L_2 = (\bigcup_{1 \leq i \leq n} f_i^{m_i} (f_i^{k_i})^*) \cup F$ where $f_i \in Q$, $m_i \geq 3$, $k_i \geq 1$ $(1 \leq i \leq r)$ and $F \subseteq X^*$ is a finite set.

From the above, we have :

Theorem 2. *For any $i \geq 3$, there is no infinite context-free language $L \subseteq X^*$ such that $L \subseteq Q^{(i)}$.*

As for context-free languages in $Q^{(2)}$, we have :

Theorem 3. *Let $L \subseteq X^*$ be a context-free language such that $L \subseteq Q^{(2)}$. Then $L = (\cup_{1 \leq i \leq r}(f_i\, g_i^{k_i}(g_i)^* h_i)^2) \cup F$ where $k_i \geq 1$, $f_i\, g_i^{k_i}(g_i)^* h_i \subseteq Q$ $(1 \leq i \leq r)$ and $F \subseteq X^*$ is a finite set.*

This result induces the following :

Theorem 4. *There is no dense context-free language $L \subseteq X^*$ such that $L \subseteq Q^{(2)}$.*

For a dense regular language $L \subseteq X^*$, we know that $L \cap Q$ becomes a disjunctive language. However, in the case of dense context-free languages, only the following result holds.

Theorem 5. *Let $L \subseteq X^*$ be a dense context-free language. Then $L \cap Q$ is a dense language.*

Problem 1. *Is $L \cap Q$ a disjunctive language when $L \subseteq X^*$ is a disjunctive language ?*