数理解析研究所講究録
第 698 巻 1989 年 36-50

Bounds for Global Solutions of Some Semilinear Parabolic Equations

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1. Introduction. In this note we are concerned with the asymptotic behavior of global solutions of the initial boundary value problem $(E)_N$ (or $(E)_D$) for the semilinear parabolic equation :

$$(E)\quad \begin{cases} (1)\quad u_t(x,t) - \Delta u(x,t) = f(x,u(x,t)),\ (x,t)\in\Omega\times[0,\infty), \\ (2)\quad u(x,0) = u_0(x),\quad x\in\Omega, \end{cases}$$

with the Neumann boundary condition

$$(3)\quad \partial u(x,t)/\partial n = 0,\quad (x,t)\in\partial\Omega\times[0,\infty),$$

(or the Dirichlet boundary condition

$$(4)\quad u(x,t) = 0,\quad (x,t)\in\partial\Omega\times[0,\infty).\ )$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and f is a continuous function from $\Omega\times\mathbb{R}^1$ to $\mathbb{R}^1$. It is well known that if f is superlinear in u, then (E) has solutions which blow up in finite time. So it would be natural to ask whether (E) has a global solution which blows up at ∞ or not.
This kind of problem was first studied by [7,8] for an abstract equation of the form $u_t + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) = 0$ in a real Hilbert space H, where $\partial\varphi^i$ are subdifferentials of lower semicontinuous convex and homogeneous functionals φ^i (i=1,2) on H. As an application of a result of [7], it is shown in [8] that

every global solution of $(E)_D$ with $f(x,u) = |u|^{p-2}u$ is uniformly bounded in $H_o^1(\Omega)$ with respect to time t, provided that $p < 2^*$, $2^* = \infty$ for $N=1,2$; $2^* = 2N/(N-2)$ for $N \geqq 3$. Ni-Sacks-Tavantzis [6] also studied $(E)_D$ for the case where Ω is convex and $f(x,u) = |u|^{p-2}u$ and showed that if $2 < p < 2 + 2/N$, then every positive global solution of $(E)_D$ is uniformly bounded in $L^\infty(\Omega)$ with respect to t and that if $2^* \leqq p$ $(N \geqq 3)$, then $(E)_D$ has a global solution u such that $|u(t)|_{L^\infty} \longrightarrow \infty$ as $t \to +\infty$.

Cazenave-Lions [2] treated more general nonlinear terms $f(x,u) = f_o(u)$ satisfying

$$(f)_o \begin{cases} (o) & f_o \in C^1(\mathbb{R}^1;\mathbb{R}^1), \\ (i) & |f_o(u)| \leqq C_1|u| + C_2|u|^{p-1} \ \forall u \in \mathbb{R}^1, \ p < 2^*, \\ (ii) & u f_o(u) \geqq (2+\varepsilon) F_o(u), \ \forall u \in \mathbb{R}^1, \ \varepsilon > 0, \ F_o(u) = \int_o^u f_o(t)\,dt, \end{cases}$$

and showed that every global solution of $(E)_D$ is bounded in $L^\infty(\Omega)$ uniformly in $[t,\infty)$ for any $t > 0$. Furthermore this bound depends only on t and the $H_o^1(\Omega)$-norm of u_o, provided that $2 < p < 2_*$, $2_* = \infty$ for $N = 1$; $2_* = 2 + 12/(3N-4)$ for $N \geqq 2$. Giga [4] removed this resteiction on p for positive solutions, i.e.,he showed that if $p < 2^*$, then the $L^\infty(\Omega)$ bound for every positive global solution depends only on the $L^\infty(\Omega)$-norm of u_o.

For all these studies, it seems that there is no result for the Neumann problem $(E)_N$ in this direction in spite of its impartance. In studying $(E)_N$, it must be noted that the methods in [2,4,6] rely much on the Dirichet boundary condition, so do not work for $(E)_N$.

The main purpose of this note is to show that " phase plane " method introduced in [7,8] works also for $(E)_N$. However, in order to apply this method to $(E)_N$, we have to remove two major restrictions in [7,8], i.e., conditions that $\partial\varphi^1$ is coercive and $\partial\varphi^2$ is homogeneous. As a matter of course, in carrying out this, we need much more careful consideration in the phase space than in [7,8]. In this note we are concerned with the following nonlinearity of f :

$$
(f) \quad \begin{cases} f(\cdot,\cdot) \text{ is a continuous function from } \Omega\times\mathbb{R}^1 \text{ into } \mathbb{R}^1 \\ \text{and there exist constants } K_i\ (i = 0,1,2,3) \text{ and numbers} \\ p \in (2, 2^*),\ \delta > 0 \text{ and } \varepsilon > 0 \text{ such that} \\ \text{(i)} \quad |f(x,u)| \leqq K_0\,(1 + |u|^{p-1}) \quad \forall (x,u) \in \Omega\times\mathbb{R}^1, \\ \text{(ii)} \quad F(x,u) = \int_0^u f(x,t)\,dt \geqq K_1\,|u|^{2+\delta} - K_2 \quad \forall (x,u) \in \Omega\times\mathbb{R}^1, \\ \text{(iii)} \quad u\,f(x,u) \geqq (2+\varepsilon)\,F(x,u) - K_3 \quad \forall (x,u) \in \Omega\times\mathbb{R}^1. \end{cases}
$$

Then our main results are stated as follows.

<u>Theorem I</u>. Let (f) be satisfied and u be a global solution of $(E)_N$ (or $(E)_D$) such that $u \in V \equiv W^{1,2}_{loc}([0,\infty); L^2(\Omega)) \cap L^2_{loc}([0,\infty); H^2(\Omega))$. Then there exists a positive constant $C_0 = C_0(|u_0|_{H^1}, K_0, K_1, K_2, K_3, \delta, \varepsilon, |\Omega|)$ such that

(5) $\displaystyle\sup_{t \geqq 0} |u(t)|_{L^2} \leqq C_0$,

(6) $\displaystyle\sup_{t \geqq 0} |u(t)|_{H^1} < +\infty$,

(7) There exists a number T_1 such that $\displaystyle\sup_{t \geqq T_1} |u(t)|_{H^1} \leqq C_0$,

(8) $\displaystyle\sup_{t \geqq 0} |u(t)|_{H^1} \leqq C_0$, provided that $p \in (2, 2_*)$, $2_* = \infty$ for $N = 1$ and $2_* = 2 + 12/(3N - 4)$.

<u>Theorem II</u>. Let (f) be satisfied and u be a global solution of $(E)_N$ (or $(E)_D$) such that $u_o \in L^\infty(\Omega)$ and $u \in L_{loc}([0,\infty); L^\infty(\Omega)) \cap W^{1,2}_{loc}((0,\infty); L^2(\Omega)) \cap L^2_{loc}((0,\infty); H^2(\Omega))$. Then there exists a positive constant $C_1 = C_1(|u_o|_{L^\infty}, K_o, K_1, K_2, K_3, \delta, \varepsilon, |\Omega|)$ such that

(9) $\sup_{t \geqq 0} |u(t)|_{L^\infty} < \infty$,

(10) There exists a number T_1 such that $\sup_{t \geqq T_1} |u(t)|_{L^\infty} \leqq C_1$,

(11) $\sup_{t \geqq 0} |u(t)|_{L^\infty} \leqq C_1$, provided that $p \in (2, 2_*)$.

<u>Remark 1</u>. (1) Assertions in Theorems I and II hold true also for Robin problem $(E)_R$, i.e., (E) with more general boundary condition :

$$(3)' \quad \frac{\partial u}{\partial n}(x,t) + \sigma(x)\, u(x,t) = 0, \ (x,t) \in \partial\Omega \times [0,\infty), \ \sigma \in L^\infty(\partial\Omega).$$

The following arguments for $(E)_N$ can be applied for $(E)_R$ with slight modifications.

(2) Conditions $(f)_o$ does not allow f to contain linear or sublinear parts, but condition (f) allow it. For example, $f(x,u) = |u|^{q-2}u + |u|^{p-2}u$ with $1 < q \leqq 2 < p$ satisfies (f) but not $(f)_o$.

(3) If f satisfies condition (f), then so does $g(x,u) = f(x,u) + u$. Indeed it is clear that g satisfies (o)-(ii) of (f), and since

$$u\,g(x,u) = u^2 + u\,f(x,u) \geqq u^2 + (2+\varepsilon)\,F(x,u) - K_3,$$

$$(2 + \varepsilon/2)(u^2/2 + F(x,u)) + \varepsilon F(x,u)/2 - \varepsilon u^2/4 - K_3,$$

$$(2 + \varepsilon/2)\int_0^u g(x,t)\,dt + \varepsilon(K_1|u|^{2+\delta} - K_2 - u^2/2)/2 - K_3,$$

and $K_1 |u|^{2+\delta} - K_2 - |u|^2/2$ is bounded below, $g(x,u)$ also satisfies (iii) of (f) with ε and K_3 replaced by $\varepsilon/2$ and some K_3'.

2. Proofs of Theorems. We shall give here proofs only for Neumann problem $(E)_N$, which are more complicated than those for $(E)_D$ and also valid for $(E)_D$.

Instead of (1), let us here consider its equivalent :

$$(1)' \qquad u_t - \Delta u + u = u + f(x,u) = g(x,u) \qquad (x,t) \in \Omega \times [0,\infty).$$

As was seen in (3) of Remark 1, g also satisfies (f). We use the same K_i, δ and ε for g as before. In what follows, $(\cdot,\cdot)$ and $|\cdot|$ denote the inner product and norm of $L^2(\Omega)$. We also denote by $|\cdot|_r$ and $\|\cdot\|$ the $L^r(\Omega)$ norm and $H^1(\Omega)$ norm respectively. Now we introduce several functionals on $H^1(\Omega)$

- $A(u) = \frac{1}{2}(|\nabla u|^2 + |u|^2) = \frac{1}{2}\|u\|^2$
- $G(u) = \int_\Omega \int_0^{u(x)} g(x,t)\, dt\, dx$,
- $J(u) = A(u) - G(u)$,
- $j(u) = (g(\cdot,u(\cdot)), u(\cdot)) - 2A(u)$

and subsets of $H^1(\Omega)$:

- $S_a = \{ u \in H^1(\Omega) ; j(u) = a \}$, $a \in R^1$.

Then, by virtue of (ii) and (iii) of (f), we obtain

$$(12) \qquad G(u) \geq K_1 |u|_{2+\delta}^{2+\delta} - K_2 |\Omega| \qquad \forall u \in H^1(\Omega), \ (|\Omega| \text{ is the volume of } \Omega.)$$

(13) $\quad j(u) \geqq \varepsilon G(u) - 2J(u) - K_3 \quad \forall u \in H^1(\Omega),$

(14) $\quad A(u) \geqq (1 + \varepsilon/2)\, G(u) - K_3/2 - a/2 \quad \forall u \in S_a .$

Furthermore, by (i) of (f) and Sobolev's inequality : $|u|_p \leqq C_b \|u\|$, there exists a constant d_o such that $d_o \geqq K_3$ and $G(u) \leqq d_o (A(u)^{p/2} + 1)$, whence follows

(15) $\quad A(u) \geqq ([G(u)/d_o - 1]^+)^{2/p}, \quad$ where $\quad [a]^+ = \max(a, 0).$

(Note that d_o is a constant depending only on $K_o, K_3, p, |\Omega|$ and embedding constant C_b.) Introducing a new parameter $\alpha = (2 + \varepsilon)\, d_o - K_3 > 0$ and taking account of (14) and (15), we can draw the following Fig. 1 which illustrates how S_o, S_α and lines $J(u) = J(u_o)$ and $J(u) = -d_o$ are located in the $(G(u), A(u))$-phase plane.

We here claim the following proposition.

<u>Proposition 1</u>. Let u be a global solution of $(E)_N$ belonging to V. Then we have

(i) $\quad J(u(t))$ is monotone decreasing in t,

(ii) $\quad J(u(t)) \geqq -d_o \quad \forall t \geqq 0,$

(iii) $\quad \int_o^\infty |u_t(t)|^2 dt \leqq J(u_o) + d_o,$

(iv) $\quad |u(t)| \leqq C_o \quad \forall t \geqq 0,$

(v) $\quad \int_t^{t+1} G(u(s))^2 ds \leqq C_o \quad \forall t \geqq 0,$

(vi) $\quad \int_t^{t+1} \|u(s)\|^4 ds \leqq C_o \quad \forall t \geqq 0,$

where C_o is a constant depending only on K_i ($i = 0,1,2,3$), ε, δ, d_o and $J(u_o)$.

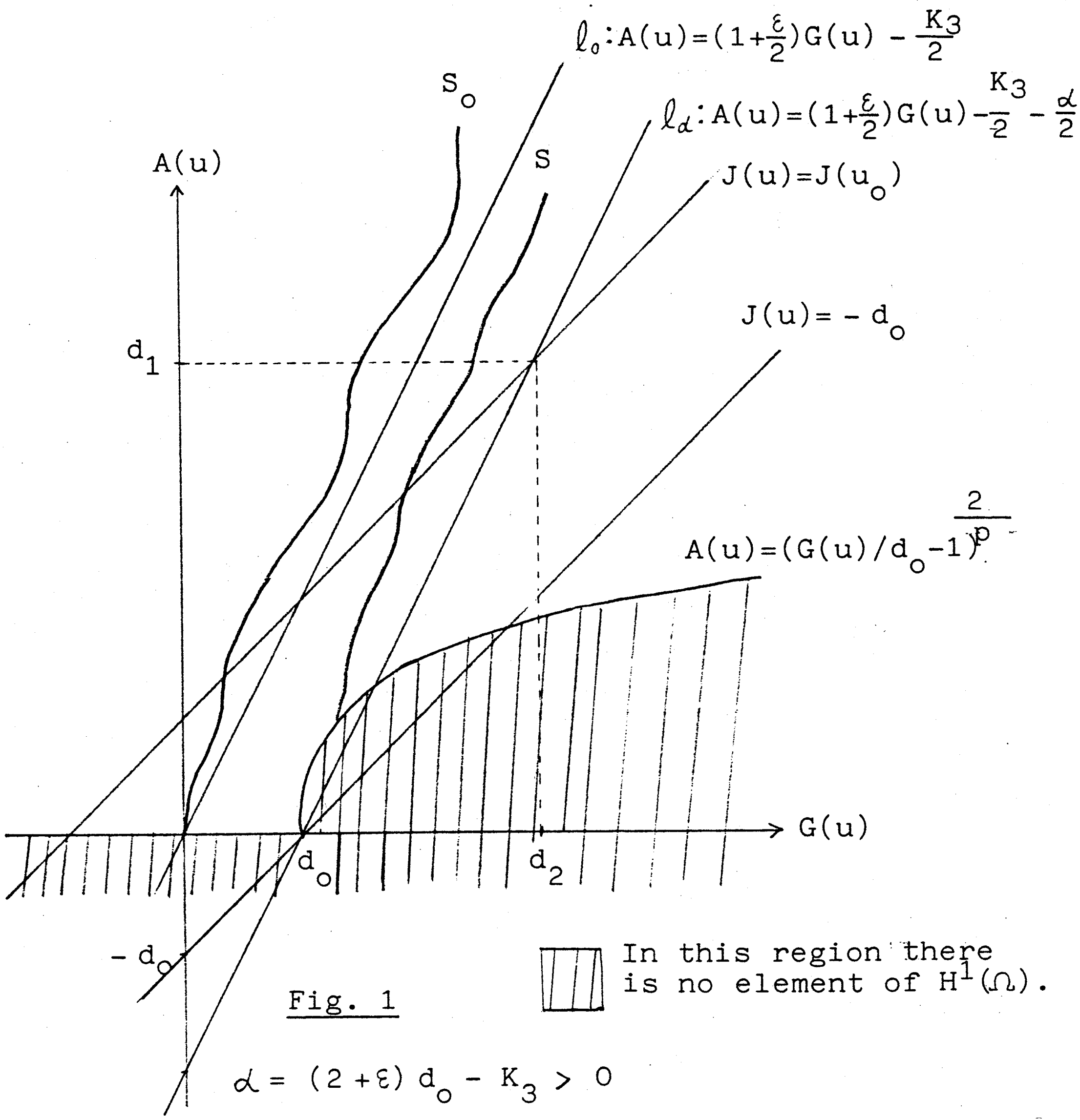

Fig. 1

$\alpha = (2+\varepsilon)\, d_o - K_3 > 0$

$d_1 = (2+\varepsilon)(J(u_o) + d_o)/\varepsilon$

$d_2 = (2\, J(u_o) + (2+\varepsilon)\, d_o)/\varepsilon$

$d_o = d_o(K_o, K_3, p, |\Omega|, C_b)$

Proof of Proposition 1. Multiplying (1)' by u_t and u, we have

$$(16)\qquad |u_t(t)|^2 + \frac{d}{dt} J(u(t)) = 0 \quad \text{for a.e. } t \in [0,\infty),$$

$$(17)\qquad \frac{1}{2}\frac{d}{dt}|u(t)|^2 = j(u(t)) \quad \text{for a.e. } t \in [0,\infty).$$

Then assertion (i) is a direct consequence of (16), and it follows from (12),(13),(17) and (i) that

$$(18)\qquad \frac{1}{2}\frac{d}{dt}|u(t)|^2 \geqq \varepsilon G(u(t)) - 2J(u_o) - K_3$$

$$\geqq \varepsilon K_1 |\Omega|^{-\delta/2} |u(t)|^{2+\delta} - \varepsilon K_2 |\Omega| - 2J(u_o) - K_3.$$

Suppose that $J(u(t_o)) < -d_o$ for some t_o, then in view of (i) and Fig. 1, we find that $j(u(t)) \geqq \alpha > 0$ for all $t \geqq t_o$. Hence (17) and (18) assure that there exists $t_1 > 0$ such that

$$(19)\qquad \frac{d}{dt}|u(t)|^2 \geqq \varepsilon K_1 |\Omega|^{-\delta/2} |u(t)|^{2+\delta} \quad \text{for a.e. } t \geqq t_1,$$

which implies that $|u(t)|$ blows up in a finite time. This is a contradiction. Thus (ii) is verified. Consequently integration of (16) over $[0,\infty)$ gives (iii). Suppose now that there exists a $t_1 > 0$ such that

$$\varepsilon K_1 |\Omega|^{-\delta/2} |u(t_1)|^{2+\delta} / 2 \geqq \varepsilon K_2 |\Omega| + 2J(u_o) + K_3,$$

then by (18), $|u(t)|$ is monotone increasing in the neibourhood of t_1. Consequently (19) holds for a.e. $t \geqq t_1$, this again leads to a contradiction. Thus we obtain a priori bound :

$$(20)\ \sup_{t \geqq 0} |u(t)| \leqq K_4 \equiv [\{4\varepsilon K_2 |\Omega| + 4J(u_o) + 2K_3\} |\Omega|^{\delta/2} / \varepsilon K_1]^{1/(2+\delta)}$$

Since $d|u(t)|^2/dt = 2(u(t), u_t(t)) \leqq 2|u(t)|\,|u_t(t)|$, assertion (v) is derived from (iii),(iv) and integration of (18) over $[t, t+1]$. Assertion (vi) follows from (v) and the fact that $J(u(t)) \leqq J(u_o)$ or $A(u(t)) \leqq B(u(t)) + J(u_o)$. Q.E.D.

Before proceeding to the proof of Theorem I, we prepare the following lemmas.

Lemma 1. Let $r, q \in [1,\infty]$ and m be a non-negative integer. Then

$$|u|_s \leqq C\, |u|^a_{W^{m,r}}\, |u|^{1-a}_q \quad \forall u \in W^{m,r}(\Omega) \cap L^q(\Omega)$$

holds for any numbers $a \in [0,1]$ and $s \in [1,\infty]$ satisfying

$$1/s = a(1/r - m/N) + (1-a)/q,$$

where C is a constant depending only on Ω, r, q, m and a.

For a proof of this lemma, see Friedmann [3].

Lemma 2. Let (i) of (f) be satisfied and u be a global solution of $(E)_N$ belonging to V. Then there exists a positive monotone decreasing function $T(\cdot)$ such that

(21) $\quad \|u(t)\| \leqq \|u(t_o)\| + 1$ for all t_o and $t \in [t_o, t_o+T(\|u(t_o)\|)]$.

Proof. First of all, we note that there exists a number $\lambda \in (0,2]$ such that

(22) $$|u|^{2(p-1)}_{2(p-1)} \leqq C\, |u|^{2-\lambda}_{H^2}\, \|u\|^{2p-4+\lambda} \quad \forall u \in H^2(\Omega).$$

Indeed, for the case $N = 1, 2$ or $N \geq 3$ and $2(p-1) \leqq 2N/(N-2)$, we can take $\lambda = 2$. For the other case, we have only to apply Lemma 1 with $s = 2(p-1)$, $m = r = 2$ and $q = 2N/(N-2)$ and use the fact that $H^1(\Omega)$ is continuously embedded in $L^{2N/(N-2)}(\Omega)$. This inequality implies that there exists a monotone increasing function $M(\cdot)$ such that

$$(23) \qquad |g(\cdot,u)|^2 \leqq \frac{1}{2}|\Delta u|^2 + M(\|u\|) \quad \forall u \in H^2(\Omega).$$

Multiplying (1)' by $-\Delta u(t) + u(t)$, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + |\Delta u(t)|^2 + |u(t)|^2 + 2|\nabla u(t)|^2 \leqq |g(\cdot,u)||\Delta u(t)|.$$

Then, by (23), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 \leqq M(\|u(t)\|) \quad \text{for a.e. } t \in [0,\infty),$$

by which we can easily verify (21) by taking $T(r) = 1/2M(r+1)$.

Q.E.D.

Now we proceed to the proof of Theorem I.

Proof of (6) and (7). By virtue of (iii) of Proposition 1, there exists a positive number T_o such that

$$(24) \qquad \int_{T_o}^{\infty} |u_t(t)|^2\,dt \leqq \left(\frac{\alpha}{K_4}\right)^2 T(d_1).$$

Then we have

$$(25) \qquad \|u(t)\| \leqq d_1 + 1 \quad \text{for all} \quad t \geqq T_1 \equiv T_o + \frac{K_4^2}{2\alpha}.$$

Suppose that this does not hold. Then there exists a $t_1 \geqq T_1$ such that $\|u(t_1)\| > d_1 + 1$. Hence, by Fig. 1, $j(u(t_1)) > \alpha$. Therefore there exists a $t_o < t_1$ such that $j(u(t_o)) = \alpha$ and $j(u(t)) > \alpha$ for all $t \in (t_o, t_1]$. Then integration of (17) over $[t_o, t_1]$ and (20) give $t_1 - t_o \leqq K_4^2/2\alpha$, i.e., $t_o \geqq T_o$. Again integrating (17) over $[t_o, t_1]$, we find

$$\alpha(t_1 - t_o) \leqq \int_{t_o}^{t_1} j(u(t))\,dt \leqq \int_{t_o}^{t_1} |u_t(t)||u(t)|\,dt$$
$$\leqq K_4(t_1 - t_o)^{1/2}\left(\int_{T_o}^{\infty}|u_t(t)|^2 dt\right)^{1/2}.$$

Hence, by (24), $t_1 - t_o \leqq T(d_1)$. Since $\|u(t_o)\| \leqq d_1$, Lemma 2 assures that $\|u(t_1)\| \leqq d_1 + 1$. This is a contradiction. Thus (5)-(7) of Theorem I are verified.

Proof of (8). As was seen above, $u(t)$ can not stay in the region $\{u \in H^1(\Omega)\,;\, j(u) > \alpha\}$ longer than $\gamma \equiv K_4^2/2\alpha$. Then for any $t \geqq 0$, there exists $t_o \in [\max(0, t-\gamma), t]$ such that $\|u(t_o)\| \leqq d_3 \equiv \max(\|u_o\|, d_1)$. Therefore, in order to prove (8), it suffices to show that $\|u(t)\| \leqq C_o$ for all $t \in I \equiv [t_o, t_o+\gamma]$. For this purpose, we prepare several results on u.

Lemma 3. If $u \in L^r(I\,;\,L^r(\Omega))$, $r < 2^*$, then

$$|u|_{L^\infty(I;L^{r/2+1}(\Omega))} \leqq C(|u|_{L^r(I;L^r(\Omega))}, d_o, d_3, r).$$

Proof. We note that

$$\frac{1}{s}\frac{d}{dt}|u(t)|_s^s = (u_t(t), |u|^{s-2}u(t)) \leqq |u_t(t)||u(t)|_r^{s-1}$$
$$\leqq \frac{1}{2}(|u_t(t)|^2 + |u(t)|_r^r),\ s = \frac{r}{2}+1.$$

Then integration of this over I and (iii) of Proposition 1 assure the assertion. Q.E.D.

Proposition 2. If $u \in L^\infty(I\,;\,L^r(\Omega))$ with $N(p-2)/2 < r < 2^*$, then

$$|u|_{L^\infty(I;H^1(\Omega))} \leqq C(|u_o|_{L^\infty(I;L^r(\Omega))}, C_o).$$

<u>Proof</u>. Multiplying (1)' by $|u|^{\ell-2}u(t)$ $(2 < \ell < 2^*)$, we obtain, by (i) of (f),

$$(26)\quad \frac{1}{\ell}\frac{d}{dt}|u(t)|_\ell^\ell + \frac{4(\ell-1)}{\ell^2}\| |u(t)|^{\ell/2}\|^2 \leqq K_o(2|u(t)|_{p+\ell-2}^{p+\ell-2} + |\Omega|).$$

For all $\ell > (p-2)N/2$, Lemma 1 with $s = 2(p+\ell-2)/\ell$, $m = 1$, $r = q = 2$ and u replaced by $|u|^{\ell/2}$ gives

$$(27)\quad |u|_{p+\ell-2}^{p+\ell-2} \leqq C|u|^{p+\lambda\ell-2}\| |u|^{\ell/2}\|^{2-\lambda}, \quad \lambda > 0.$$

Then, by integration of (26) over I and (27) with $\ell = r$, we get

$$(28)\quad | |u|^{r/2}|_{L^2(I;H^1(\Omega))} \leqq C(|u|_{L^\infty(I;L^r(\Omega))}, \|u_o\|)$$

Again by Lemma 1 with $s = 2(N+2)/N$, $m = 1$, $r = q = 2$ and u replaced by $|u|^{r/2}$, we now deduce

$$(29)\quad |u|_{(2+N)r/N}^{(2+N)r/N} \leqq C|u|_r^{2/N}\| |u|^{r/2}\|^2.$$

Then it follows from (28) and (29) that $u \in L^{(2+N)r/N}(I;L^{(2+N)r/N}(\Omega))$. Hence, by Lemma 3, $u \in L^\infty(I; L^{(N+2)r/2N+1}(\Omega))$. Repeating this procedure, we observe that $u \in L^\infty(I;L^{r_i}(\Omega))$, where r_i are defined by the recurrence formula

$$r_{i+1} = \frac{N+2}{2N} r_i + 1, \; i = 1,2,\cdots, \; r_1 = r.$$

Since $r_i \longrightarrow 2^*$ as $i \longrightarrow \infty$ and $p < 2^*$, we can show by finite steps that $|u|_{L^\infty(I;L^p(\Omega))} \leqq C(|u|_{L^\infty(I;L^r(\Omega))}, C_o)$. Thus the assertion follows from (ii) of Proposition 1. Q.E.D.

Proposition 3. For $N = 1,2,3$ or 4, we have

$$|u|_{L^\infty(I;L^q(\Omega))} \leqq C_o \qquad \text{for all} \quad q < q^*,$$

where $q^* = \infty$ for $N = 1$ and $q^* = 2 + 8/(3N-4)$ for $N = 2,3,4$.

Proof. Let $q_1 = 2$ and $q_{i+1} = (4-N)q_i/2N + (6N+8)/2N$, $i = 1,2,\cdots$, and apply Lemma 1 with $s = q_{i+1}$, $m = 1$, $r = 2$ and $q = (2+q_i)/2$. Then we have

$$|u|_{q_{i+1}}^{q_{i+1}} \leqq C\,|u|_{(2+q_i)/2}^{q_{i+1}-4}\,\|u\|^4.$$

Therefore, by virtue of (vi) of Proposition 1 and Lemma 3, $u \in L^{q_i}(I;L^{q_i}(\Omega))$ for all q_i. Since $q_i \to q^*$ as $i \to \infty$, we can derive the assertion by finite steps. Q.E.D.

Proposition 4. For $N \geqq 5$, we have

$$|u|_{L^\infty(I;L^q(\Omega))} \leqq C_o \qquad \text{for all} \quad q < q_* = 3 + (N-4)(2-p)/4.$$

Proof. Since $p < 2^* \leqq 4$ for $N \geqq 4$ and (vi) of Proposition 1 assures that $|u|_{L^4(I;L^{2^*}(\Omega))} \leqq C_o$, we get $|u|_{L^p(I;L^p(\Omega))} \leqq C_o$. Let $p_1 = p$, $p_{i+1} = (N-4)p_i/(N-2) + [8+(N-4)(2-p)]/(N-2)$, and $s_i = 2 + p_i - p$, $i = 1,2,\cdots$. Suppose that $u \in L^{p_i}(I;L^{p_i}(\Omega))$, then by (26) with $\ell = s_i$, $|u|^{s_i/2} \in L^2(I;H^1(\Omega))$. Moreover Lemma 1 with $s = 2p_{i+1}/s_i$, $m = 1$, $r = 2$, $q = 2\cdot 2^*/s_i$ and u replaced by $|u|^{s_i/2}$ yields

$$|u|_{p_{i+1}}^{p_{i+1}} \leqq C\,|u|_{2^*}^{\lambda_1}\,\|\,|u|^{s_i/2}\|^{\lambda_2} \qquad \text{with} \quad \lambda_1/2 + \lambda_2 \leqq 2,$$

whence follows $|u|_{L^{p_{i+1}}(I;L^{p_{i+1}}(\Omega))} \leqq C_o$. Since $p_i \to q_*$ as $i \to \infty$, the assertion can be derived by finite steps. Q.E.D.

In order to prove (8), we have only to combine Propositions 2 with 3 (for the case $N \leqq 4$) and 2 with 4 (for the case $N \geqq 5$). In fact, $N(p-2)/2 < q^*$ or $N(p-2)/2 < q_*$ holds if and only if $p < 2_*$. Q.E.D.

Proof of Theorem II. We shall rely on Moser's iteration scheme to obtain L^∞ bound via H^1 bound. The following lemma plays an important role in this procedure.

Lemma 4. Let $w \in W^{1,1}_{loc}([0,\infty);L^2(\Omega)) \cap L^\infty_{loc}([0,\infty);L^\infty(\Omega) \cap H^1(\Omega))$ satisfy

(30) $$\frac{d}{dt}|w(t)|_r^r + C_1 r^{-\theta_1} \| |w(t)|^{r/2} \|^2 \leqq C_2 r^{\theta_2}(|w(t)|_r^r + 1) \quad \text{a.e. } t \in [0,\infty)$$

for all $r \geqq 2$, where $C_1 > 0$ and $C_2, \theta_1, \theta_2 \geqq 0$.
Then there exist constants a,b,c,d such that

$$\sup_{t \geqq 0} |w(t)|_\infty \leqq a\, 2^{\theta_2 + (\theta_1 + \theta_2) b} M_0 ,$$

where $M_0 = \max\left(1,\ c|u_0|_\infty,\ \sup_{t \geqq 0} |w(t)|^d\right)$.

Proof. When w belongs to $H^1_0(\Omega)$ for a.e. t, this is proved by Nakao [5] (See Lemma 3.1). By using Lemma 1 instead of Gagliado-Nirenberg's inequality, one can prove this lemma by the same argument as in [5]. Q.E.D.

Put $\lambda_1 = r(1 - N(p-2)/2p)$, $\lambda_2 = p - 2$, $\lambda_3 = Nr(p-2)/2p$ and $\theta = Nr/(N-2)$. Then, by Hölder's inequality, we get

$$|u|_{p+r-2}^{p+r-2} \leqq |u|_r^{\lambda_1} |u|_p^{\lambda_2} |u|_\theta^{\lambda_3}.$$

Applying Sobolev's embedding theorem and Young's inequality, we deduce

$$
(31)\quad \begin{aligned} |u(t)|_{p+r-2}^{p+r-2} &\leqq C\,|u(t)|_r^{\lambda_1} \| \, |u(t)|^{r/2} \|^{2\lambda_3/r} \\ &\leqq \frac{2(r-1)}{r^2} \| \, |u(t)|^{r/2} \|^2 + C\, r^{\frac{N(p-2)}{2p-(p-2)N}} |u(t)|_r^r, \end{aligned}
$$

where C is a general constant depending on $\sup\{|u(t)|_p;\ t \geqq 0\}$. Then (31) and (26) with $\ell = r$ imply that $u(t)$ satisfies (30) with $C_1 = 1$, $\theta_1 = 0$ and $\theta_2 = 2p/(2p-(p-2)N)$. Thus (9) is verified by Lemma 4.

On the other hand, it is easy to show that there exists a positive number T_0 depending only on $|u_0|_\infty$ such that $|u(t)|_\infty \leqq |u_0|_\infty + 1$ for all $t \in [0, 2T_0]$ and $\|u(T_0)\| \leqq C(|u_0|_\infty)/T_0$. Hence, (10) and (11) follows from (7) and (8) respectively.

Q.E.D.

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