数理解析研究所講究録
第 701 巻 1989 年 1-18

Nonstandard Analysis and Applications to Probability Theory and Potential Theory

Peter A. Loeb,

Department of Mathematics, Science University of Tokyo, Noda 278
Department of Mathematics, University of Illinois,
1409 West Green Street, Urbana, Illinois, 61801, USA.

The notion of an infinitesimal has been used since the time of Archimedes. His book, "The Method", was discovered in 1909 in Library of Constantinoble. Calculus was formulated using infinitesimals in the late sixteen hundreds by the German mathematician Leibniz and by Newton. Leibniz regarded infinitesimals as a useful fiction which facilitated mathematical computation and invention. Even then, there was controversy. Bishop George Berkeley wrote, "What are these fluxions? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?" His point was to question the intellectual consistency of atheists who believed in infinitesimals. The question was, how can there be a positive number which is smaller than any real number without being zero.

Abraham Robinson in 1960 (see [31]) gave a rigorous foundation for the use of infinitesimals in analysis. He used model theory, a branch of mathematical logic. Robinson's invention, called nonstandard analysis, is more than a justification of the method of infinitesimals; it is a powerful new tool for mathematical research. In the 25 years since Robinson's discovery, the use of nonstandard models has led to many new insights and some solutions to unsolved problems in areas as diverse as functional analysis, probability theory, complex function theory, potential theory, number theory, mathematical physics, and mathematical economics.

As a simple introduction, we will extend the real numbers $\mathbb{R}$ with an ordered field $^*\mathbb{R}$ containing infinitesimals.

Here is a simple construction of $^*\mathbb{R}$ from real–valued sequences:

Sequences do not form a field. E.G.: E = evens O = odds in $\mathbb{N}$ then $\chi_E \cdot \chi_O$ is identically 0.

Fix a FREE ULTRAFILTER $\mathcal{U}$ in $\mathbb{N}$.

$\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ and $\phi \notin \mathcal{U}$,

$A \in \mathcal{U}$ & $B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$

$A \subset \mathbb{N}$ & $A \notin \mathcal{U} \Rightarrow \mathbb{N} - A \in \mathcal{U}$

A finite in $\mathbb{N} \Rightarrow \mathbb{N} - A \in \mathcal{U}$

We say a property holds a.e. if it holds on some set $U \in \mathcal{U}$

We set a sequence $<r_i> \equiv <s_i>$ when $r_i = s_i$ a.e. $^*\mathbb{R}$ = the set of equivalence classes.

Now $\chi_E \equiv 0$ or $\chi_O \equiv 0$. $\mathbb{R} \subset {}^*\mathbb{R}$ via map $c \to [<c>]$. The sequence $<1/i>$ represents an infinitesimal, and $<i>$ represents an infinite number. In general, A property holds for $^*\mathbb{R}$ if it holds a.e. on $\mathbb{N}$.

An **internal subset** E OF $^*\mathbb{R}$ corresponds to a sequence $<A_i>$ by the relation $[<r_i>] \in E$ iff $r_i \in A_i$ a.e. A non–internal set is called **external**. For example, $\mathbb{N}$ is external in $^*\mathbb{R}$. A **hyperfinite set** is such an internal set E with A_i finite a.e. E.G., $A_i = \{1, 2, \cdots, i\}$. The "internal cardinality" of this particular set is $[<i>]$.

It is better to ignore any particular construction of $^*\mathbb{R}$ and work with just the properties. That is what we will introduce next.

I. MODEL THEORY VIEWPOINT IN NONSTANDARD ANALYSIS.

We start with **superstructure** V(S), where S is a set containing R. V(S) = all sets obtained from S in a finite number of steps using the usual operations of set theory. E.G., V(S) contains individuals including all reals, the set of Lebesgue measurable sets, the set of all Borel measures on $\mathbb{R}$.

Next we let L be a formal language for V(S). L contains the following: A name for each object in V(S), variables, connectives, quantifiers, brackets, and sentences built from these.

1.1 THEOREM (Robinson): There is a (not unique) structure $V({}^*S)$ built from a set ${}^*S \supset S$ such that:

1) A name of an object in V(S) names a similar object in $V({}^*S)$ i.e., one built in the same way. (We write ${}^*A$ for the object in $V({}^*S)$ with the same name as A. The set A is called standard, ${}^*A$ is called the (nonstandard) extension of A. The extension of any individual s is denoted by s not ${}^*s$.)

2) (Transfer Principle): Every sentence in L true for V(S) is true when interpreted in $V({}^*S)$; quantification, is over "internal" objects in $V({}^*S)$.

3) If $A \in V(S)$ is a set, $\exists$ "hyperfinite" $B \in {}^*\mathcal{P}_F(A)$ such that for each $a \in A$, ${}^*a \in B$.

Internal objects in $V({}^*S)$ are objects which are members of the extensions of standard objects. **External** means noninternal. Here is a proof that $\mathbb{N}$ is external in ${}^*\mathbb{N}$: If this were not true, then there would exist a first infinite element of ${}^*\mathbb{N}$ and thus a last element of $\mathbb{N}$.

Hyperfinite sets are internal sets in internal 1:1 correspondence with an initial segment of $^*\mathbb{N}$. Such sets have the formal combinatorial properties of finite sets. For example, let $^*\mathbb{N}_\infty$ denote the infinite elements in $^*\mathbb{N}$. For any $\eta \in {}^*\mathbb{N}_\infty$, $\mathbb{N} \subset \{n \in {}^*N: 1 \leq n \leq \eta\}$. This is an example of the Property 3.

An element $a \in {}^*\mathbb{R}$ is **finite** if $\exists\, n \in \mathbb{N}$ such that, $|a| < n$; a is **infinite** if $\forall\, n \in \mathbb{N}$, $|a| > n$; a is **infinitesimal** if $\forall\, n \in \mathbb{N}$, $|a| < 1/n$. The only infinitesimal in $\mathbb{R}$ is 0. We write $a \simeq b$ when $a - b$ is infinitesimal. The monad $m(r)$ of $r \in \mathbb{R}$ is the set $\{\rho \in {}^*\mathbb{R}: \rho \simeq r\}$.

$^*\mathbb{R}$ contains infinite positive and infinite negative elements. If ρ is finite in $^*\mathbb{R}$, there exists a unique standard real number $r \simeq \rho$. r is the supremum of the set of standard rational numbers less than ρ; r is called the standard part of ρ. We write $r = st(\rho) = {}^\circ\rho$.

Here are some example of applications in real analysis:
Let s_n be a sequence in $\mathbb{R}$. $s_n \to L$ iff $\forall\, \omega \in {}^*\mathbb{N}_\infty$, $s_\omega \simeq L$. B is a cluster point of s_n iff $\exists\, \eta \in {}^*\mathbb{N}_\infty$ such that $s_\eta \simeq B$. A real–valued function f is continuous on a set A iff $\forall\, x \in A$, $\forall\, y \in {}^*A$, $y \simeq x \Rightarrow {}^*f(y) \simeq f(x)$. f is uniformly continuous on A iff $\forall\, x \in {}^*A, \forall\, y \in {}^*A$, $y \simeq x \Rightarrow {}^*f(y) \simeq {}^*f(x)$. A set $A \subset \mathbb{R}$ is compact iff $\forall\, \rho \in {}^*A$, $\exists\, r \in A$ with $r \simeq \rho$. A function f continuous on a compact set is uniformly continuous since $\forall\, x, y \in {}^*A$ with $y \simeq x$, $r = st(x) = st(y) \in A$, so ${}^*f(y) \simeq f(r) \simeq {}^*f(x)$. A sequence s_n in a compact set A has a cluster point since $\forall\, \eta \in {}^*\mathbb{N}_\infty$, if $r = st(s_\eta)$, then $r \in A$ and $s_\eta \simeq r$. For $\Delta x \simeq 0$ but $\Delta x \neq 0$, if $st(\Delta y/\Delta x)$ is well defined, then $f'(x) = st(\Delta y/\Delta x)$. For a continuous f,

$$\int_a^b f(x)\, dx = st(\Sigma_a^b f(x)\, \Delta x).$$

Here is an application of hyperfinite sets to the proof that a continuous function f on [0,1] takes its maximum value:
Fix $\omega \in {}^*\mathbb{N}_\infty$. Let $x_0 = 0$, $x_1 = 1/\omega, \cdots, x_i = i/\omega, \cdots, x_\omega = 1$. Choose the maximum of $\{f(0), \cdots, {}^*f(x_i), \cdots, f(1)\}$; call it ${}^*f(x_k)$. Let $r = st(x_k)$. If $s \in [0,1]$, and $x_j \simeq s$, then $f(r) \simeq {}^*f(x_k) \geq {}^*f(x_j) \simeq f(s)$, so $f(r) \geq f(s)$.

II. NONSTANDARD MEASURE THEORY

One can construct a hyperfinite set X as the set of elementary outcomes in a conceptual experiment in the "nonstandard world". For coin tossing, for example, X is the set of internal sequences of 0's and 1's of length $\eta \in {}^*\mathbb{N}_\infty$. Given a hyperfinite set X, let $\mathcal{M}$ denote the set of all internal subsets of X. $\mathcal{M}$ is an internal σ–algebra but also an ordinary algebra in X. Let π denote an internal ${}^*\mathbb{R}$–valued probability measure on $(X,\mathcal{M})$. Let P be the finitely additive $\mathbb{R}$–valued measure on $(X,\mathcal{M})$ defined by setting $P(A) = st(\pi(A))$. For coin tossing, for example, $P(A) = st(|A|/2^\eta)$ where $|A|$ denotes the internal cardinality of A. Let $\sigma(\mathcal{M})$ denote the σ–algebra generated by $\mathcal{M}$.

We extend $(X,\mathcal{M},P)$ to a standard probability space $(X,\sigma(\mathcal{M}),P)$ on X as follows: First, we assume $V({}^*S)$ is "$\aleph_1$–SATURATED" (This is true for an ultrapower.) This means that any ordinary sequence $\{A_i: i \in \mathbb{N}\}$ from internal set E is an initial segment of an internal sequence $\{A_i: i \in {}^*\mathbb{N}\}$ from E. If now $\{A_i: i \in \mathbb{N}\} \subset \mathcal{M}$ is pairwise disjoint and $A = \cup A_i$ is in $\mathcal{M}$ then all but a finite number of the A_i's are empty. (If not, we can extend $\{A_i\}$ to $\{A_i: i \in {}^*\mathbb{N}\}$; the set $\{m \in {}^*\mathbb{N}: A \subset \cup_{1 \leq i \leq m} A_i\}$ is internal and contains ${}^*\mathbb{N}_\infty$, so it contains some finite element in $\mathbb{N}$.) By Carathéodory's extension theorem, P has a σ–additive extension to the external σ–algebra $\sigma(\mathcal{M})$ since, because the sum is finite, $P(A) = \Sigma\, P(A_i)$.

Given an $^*\mathbb{R}$–valued function f on X, we set $^\circ f(x) = st(f(x))$ or $\pm\infty$ if the value is infinite. A function $g: X \to \mathbb{R} \cup \{+\infty,-\infty\}$ is $\sigma(\mathcal{M})$–measurable iff $\exists$ $\mathcal{M}$–measurable f such that $^\circ f = g$ P–a.e. The function f is called a lifting of g If such an f is finite valued, then $^\circ\int f\, d\pi = \int {}^\circ f\, dP$. This is also true if $\int[|f| - (|f| \wedge \eta)]\, d\pi$ is infinitesimal for all $\eta \in {}^*\mathbb{N}_\infty$.

The above construction was used by the author in [18] to obtain an underlying space for coin tossing. Similar constructions gave Poisson processes [18] and representing measures for harmonic functions [19], [20]. The coin tossing space was used by R. M. Anderson [2] to make rigorous the probabilist's intuitive treatment of Brownian motion as an infinitesimal random walk and the Itô integral as a pathwise Stieltjes integral. In this sense, standard Brownian motion is nonstandard coin tossing. We will discuss these applications in Sections V and VI.

III. A FUNCTIONAL APPROACH TO NONSTANDARD INTEGRATION

In what follows, we will always assume $\aleph_1$–saturation. We will work with an internal set X, an internal vector lattice L consisting of $^*\mathbb{R}$–valued functions on X, and an internal positive linear functional I on L. Except for Theorem 3.9, we will assume that $1 \in L$ and $I(1)$ is finite in $^*\mathbb{R}$. This assumption corresponds to the assumption of finite measure spaces and compact topological spaces as in [25]. Without this assumption, one must assume that $1 \wedge \varphi \in L$ for each $\varphi \in L$. See [24] for this more general theory.

3.1 EXAMPLES:

1) X is a hyperfinite set, L is a set of ${}^*\mathbb{R}$–valued internal functions on X, $I(f) = (1/|X|)\cdot\Sigma f(x)$.

2) $(X, \mathcal{M}, \pi)$ is an internal probability space, L is the space of internal $\mathcal{M}$–measurable simple functions, $I(f) = \int f\, d\pi$.

3) $X = {}^*[0,1]$, $L = {}^*\mathcal{C}([0,1])$, I is a positive linear functional on L. E.G., $I(f) = {}^*\int_0^1 f(x)\, dx$.

3.2 DEFINITIONS:

L_0, the set of **null functions**, is the set of all internal and external ${}^*\mathbb{R}$–valued functions h on X such that $\forall\ \epsilon > 0$ in $\mathbb{R}\ \exists\ \varphi \in L$ with $|h| \le \varphi$ and $I(\varphi) < \epsilon$.

L_1 is the set of all real–valued functions f on X having representation $f = \varphi + h$ for some $\varphi \in L$ and $h \in L_0$.

For each $f \in L_1$, $J(f) = {}^{\circ}I(\varphi)$.

L_1 is a real vector lattice with $1 \in L_1$. The positive real–valued functional J is well defined on L_1. Note that we assume no continuity for I. With $\aleph_1$–saturation, however, we have continuity for J via the following result.

3.3 MONOTONE CONVERGENCE THEOREM:

If $\{f_n : n \in \mathbb{N}\}$ is increasing sequence in L_1 with real upper envelope F and $\sup J(f_n) < +\infty$, then $F \in L_1$ and $J(F) = \lim J(f_n)$.

PROOF. We may assume each $f_n \ge 0$. For each $n \in \mathbb{N}$, we may fix $\varphi_n \in L$ and $h_n \in L_0$ so that $f_n = \varphi_n + h_n$ and $0 \le \varphi_n \le \varphi_{n+1}$. By $\aleph_1$–saturation, $\exists\ \varphi_\omega \in L$ with $\varphi_\omega \ge \varphi_n$ for each $n \in \mathbb{N}$ and ${}^{\circ}I(\varphi_\omega) = \lim_{n\in\mathbb{N}} {}^{\circ}I(\varphi_n)$. It also follows by $\aleph_1$–saturation that $F - \varphi_\omega \in L_0$.

3.4 INTERNAL APPROXIMATION PROPERTY:

A real–valued function f is in L_1 iff $\forall \epsilon > 0$ in $\mathbb{R}$, $\exists \psi_1$ and ψ_2 in L with $\psi_1 \le f \le \psi_2$ and $I(\psi_2 - \psi_1) < \epsilon$.

PROOF of sufficiency. Assume there are sequences $\varphi_n \uparrow$ and $\psi_n \downarrow$ in L such that for each $n \in \mathbb{N}$, $\varphi_n \le f \le \psi_n$ and $I(\psi_n - \varphi_n) < 1/n$. Extend both sequences to ${}^*\mathbb{N}$. Choose $\psi_\omega \in L$ so that for each $n \in \mathbb{N}$, $\varphi_n \le \psi_\omega \le \psi_n$. Now $\varphi_n - \psi_n \le f - \psi_\omega \le \psi_n - \varphi_n$. Thus $f - \psi_\omega \in L_0$ and $f \in L_1$.

3.5 PROPOSITION. If $\varphi \in L$ is finite valued, then ${}^\circ\varphi \in L_1$ and,

$$J({}^\circ\varphi) = {}^\circ I(\varphi).$$

PROOF. ${}^\circ I(1) < +\infty$ and $\forall \epsilon > 0$ in $\mathbb{R}$, $|{}^\circ\varphi - \varphi| < \epsilon$, so ${}^\circ\varphi - \varphi \in L_0$.

3.6 DEFINITION: Fix $\mathcal{B} = \{A \subseteq X: \chi_A \in L_1\}$, $\mu(A) = J(\chi_A)\ \forall A \in \mathcal{B}$.

3.7 THEOREM. $\mathcal{B}$ is a σ–algebra and μ is a complete, σ–additive, finite measure on $(X,\mathcal{B})$. A bounded $g: X \to \mathbb{R}$ is $\mathcal{B}$–measurable iff $g \in L_1$, and then $J(g) = \int_X g\, d\mu$.

3.8 EXAMPLE: Let $(X,\mathcal{M},\pi)$ be an internal probability space and I be the π–integral on the class L of internal $\mathcal{M}$–simple functions. For $A \in \mathcal{M}$, set $P(A) = st(\pi(A))$. Then $\mathcal{M}$ is an ordinary algebra, and P is a finitely additive, real–valued measure on $\mathcal{M}$. Our construction produces a standard probability space $(X,\mathcal{B},P)$ extending $(X,\mathcal{M},P)$ with $\mathcal{M} \subset \mathcal{B}$ and $P(A) = P(A)$ for each $A \in \mathcal{M}$. If $B \in \mathcal{B}$ and $\epsilon > 0$ in $\mathbb{R}$, $\exists$ functions φ and ψ in L with $\varphi \le \chi_B \le \psi$ and $I(\psi - \varphi) < \epsilon$. Let $A_1 = \{\varphi > 0\}$, $A_2 = \{\psi \ge 1\}$. Now, $A_1, A_2 \in \mathcal{M}$, $A_1 \subseteq B \subseteq A_2$, $\pi(A_2 - A_1) \le I(\psi - \varphi) < \epsilon$. Using saturation, one shows $\exists\ C \in \mathcal{B}$ such that $\mu(C \vartriangle B) = 0$. Thus, we have direct approximation of sets in $\mathcal{B}$ by sets in $\mathcal{M}$.

In [24], the author gave a simple proof of a result from [21] stating that the inverse image under the standard part map of a compact set is measurable in any completion of the sigma algebra generated by internal Borel sets. That proof, given in the next section, is stated in terms of continuous functions, but it works just as well for simple functions. The basic fact, one that has become increasingly important in some of the recent literature, is that certain uncountable operations on internal objects yield measurable objects. To give the general principle in our context, we put the necessary finiteness condition in the statement of our result.

3.9 THEOREM. Assume κ–saturation. Let $\{\varphi_i\colon i \in I\}$ be a subset of L closed under the lattice operation $\wedge$ with $\varphi_i \geq 0$ $\forall$ i and card(I) $< \kappa$. Assume $I(\varphi_i)$ is finite for some $i \in I$ and $\bigwedge_i \varphi_i$ is finite valued. Then $f = {}^\circ(\bigwedge_i \varphi_i)$ is in L_1 and $J(f) = \inf {}^\circ I(\varphi_i)$.

Proof: By κ–saturation, there is a $\varphi \in L$ with $0 \leq \varphi \leq \varphi_i$ for all $i \in I$ and ${}^\circ I(\varphi) = \inf {}^\circ I(\varphi_i)$. For any $\epsilon > 0$ in $\mathbb{R}$ and any $i \in I$,

$$(1-\epsilon)\varphi \leq f \leq (1+\epsilon)\varphi_i,$$

and so

$$-\epsilon\cdot\varphi \leq f - \varphi \leq (1+\epsilon)\varphi_i - \varphi.$$

It follows that $f - \varphi$ is null, so $f \in L_1$ and $J(f) = {}^\circ I(\varphi)$. □

3.10 EXAMPLE. Let X be a hyperfinite set of ordinals containing all standard ordinals less than or equal to the first uncountable ordinal ω_1; if $\gamma \in X$ then $\gamma \leq \omega_1$. Let $\nu(A)$ denote the internal cardinality of A for each internal $A \subseteq X$. Even with ω_1^+ saturation, the set consisting of all elements of X which are larger than any standard countable ordinal is not in the completion of the sigma algebra generated by the internal subsets of X.

IV. FUNCTIONALS ON SPACES OF CONTINUOUS FUNCTIONS

Let Y be a standard set with a compact Hausdorff topology $\mathcal{T}$. We will work with an enlargement of a structure containing Y and $\mathbb{R}$ and assume κ–saturation with $\kappa \geq \aleph_1$ and $\kappa \geq \mathrm{Card}(\mathcal{T})$. This will be explained later. For each $y \in Y$, the monad $m(y) = \cap \{{}^*U : U \in \mathcal{T}, y \in U\}$. Each $x \in {}^*Y$ is in the monad of a unique $y \in Y$; we write $y = st(x)$. This is Robinson's criterion from [31] for the compactness of Y. With each real–valued function g on Y, associate $\tilde{g}$ on ${}^*Y$ where $\tilde{g}(x) = g(st(x))$. For $A \subset Y$, set $\tilde{A} = \cup \{m(y) : y \in A\}$ so $\chi_{\tilde{A}} = (\chi_A)^{\sim}$ on ${}^*Y$. Let $X = {}^*Y$, and let $L = {}^*\mathcal{C}(Y)$. Let I be an internal positive linear functional on $L = {}^*C(Y)$, with I(1) finite in ${}^*\mathbb{R}$. We will apply the general theory to (X,L,I). The next result is where we use κ–saturation. The proof originated in [24], and has been used above to establish the more general result, Theorem 3.9.

4.1 PROPOSITION. For each compact $K \subseteq Y$, $\tilde{K} \in \mathcal{B}$, and $\mu(\tilde{K}) = \alpha_K$, where $\alpha_K = \inf\{{}^oI({}^*f) : f \in \mathcal{C}(Y), \chi_K \leq f \leq 1\}$.

PROOF. By κ–saturation, $\exists\, \varphi \in L$ with $\chi_{{}^*K} \leq \varphi \leq \chi_{\tilde{K}}$ and ${}^oI(\varphi) = \alpha_K$. $\forall f \in \mathcal{C}(Y)$ with $\chi_K \leq f \leq 1$, $\forall \epsilon > 0$ in $\mathbb{R}$, $\varphi \leq \chi_{\tilde{K}} \leq (1+\epsilon){}^*f$, so $\chi_{\tilde{K}} - \varphi \in L_0$, $\chi_{\tilde{K}} \in L_1$, and $\mu(\tilde{K}) = J(\chi_{\tilde{K}}) = {}^oI(\varphi) = \alpha_K$.

4.2 THEOREM. Let $\mathcal{B}_Y = \{B \subseteq Y : \tilde{B} \in \mathcal{B}\}$, and $\mu_Y(B) = \mu(\tilde{B})$ $\forall B \in \mathcal{B}_Y$. Then $\mathcal{B}_Y$ is a σ–algebra containing the Borel sets, and μ_Y is a complete, regular measure. A function $g : Y \to \mathbb{R}$ is $\mathcal{B}_Y$–measurable iff $\tilde{g}$ is $\mathcal{B}$–measurable on ${}^*Y$. For $f \in \mathcal{C}(Y)$, $\int_Y f\, d\mu_Y = {}^oI({}^*f)$.

4.3 COROLLARY: The Riesz representation theorem for the dual space of $\mathcal{C}(Y)$ is now established. Just set I equal to the nonstandard extension of a standard functional on $\mathcal{C}(Y)$.

4.4 COROLLARY (Anderson–Rashid [3], Loeb [21]): If I comes from an internal Baire measure ν, Then $\mu_Y = st(\nu)$ in the weak* topology. This gives weak convergence results for measures since a weak* cluster point of a net of measures can be obtained be taking the standard part of an infinitely indexed element in the nonstandard extension of the net.

4.5 EXAMPLE: Let $Y = [0,1]$, and let I be the nonstandard extension of the standard Riemann integral on Y. A real–valued g on Y is Lebesgue integrable iff $\tilde{g} = \varphi + h$ where $\varphi \in {}^*\mathcal{C}(Y)$ and $h \in L_0$. The Lebesgue integral of g equals the standard part of the internal Riemann integral of φ. A bounded g is Riemann integrable iff $^\circ({}^*g) \in L_1$ (result with Cornea.)

4.6 EXTENSIONS: Extensions of the general theory to functions taking their values or integrals in nonstandard hulls of topological vector lattices are under development with Horst Osswald of Munich. Preliminary results can be found in [27].

V. APPLICATIONS TO PROBABILITY

POISSON PROCESSES: These are formulated in [18] as an internal random distribution of an infinite number of unit masses into an infinite number of infinitesimal intervals.

COIN TOSSING: This is formulated in [18] as follows. Fix $\eta \in {}^*\mathbb{N}_\infty$, and let $\Omega = \{-1, 1\}^\eta$. That is, Ω is the internal set of all internal tosses of length η. Let $\mathcal{M}$ be the set of internal subsets of Ω and $\pi(A) = |A|/2^\eta$. Set $P(A) = st(\pi(A))$ $\forall$ $A \in \mathcal{M}$; let P also denote the measure μ on $\mathcal{B}$ developed in the general theory. Then $(\Omega,\mathcal{B},P)$ extends $(\Omega,\mathcal{M},\pi)$ It is a probability space for coin tossing.

BROWNIAN MOTION (R. M. Anderson [2]): Take the above coin tossing space. For any coin toss $\omega \in \Omega$, let $X_i(\omega) = \omega_i$ (−1 or 1). Let $\chi(t,\cdot)$ be the internal random walk

$$\chi(t,\omega) = (1/\sqrt{\eta})\cdot\sum_{i=1}^{[\eta t]} X_i(\omega) \quad (t \in {}^*[0,1]).$$

Here, the particle located by $\chi(t,\omega)$ starts at 0 and at each time $t_i = i/\eta$, $i = 1,2,\cdot\cdot,\eta$, moves right or left by $\sqrt{\Delta t} = 1/\sqrt{\eta}$.

For P–almost every ω, $\chi(t,\omega)$ is infinitely close in the uniform topology to some standard f in $\mathcal{C}([0,1])$. For those ω and $t \in [0,1]$ Anderson set $\beta(t,\omega) = {}^\circ\chi(t,\omega)$. This is Brownian motion. The projection of P onto $\mathcal{C}([0,1])$ using the standard part map (similar to the use of the ˜ map) is Wiener measure. This gives a simple proof of Donsker's theorem.

Anderson in [2] also replaced standard functions with infinitely close functions constant on infinitesimal intervals Δt in ${}^*[0,1]$ and then integrated against $\chi(t,\omega)$. This process made Itô integrals the standard part of internal pathwise Stieltjes integrals. Anderson's work makes rigorous the formula $dx = dt^{\frac{1}{2}}$. It gives an easy proof of Itô's lemma.

VI. APPLICATIONS TO POTENTIAL THEORY

For these applications, we will work with harmonic functions on the unit disk and indicate that the results hold for much more general situations. Let D_r denote the open disk $\{z \in \mathbb{C}: |z| < r\}$, and let $D = D_1$. Let C_r be the circle $\{z \in \mathbb{C}: |z| = r\}$, and let $C = C_1$. Let $P(z,x)$ be the Poisson Kernel $(1 - |x|^2)/|z - x|^2$, and let x_0 denote the origin. The space $\mathcal{H}^1$ consisting of positive harmonic functions on D taking the value 1 at x_0 is convex and compact with respect to the topology of uniform convergence on compact subsets of D. We call this topology the ucc topology. If $f \in \mathcal{C}(C)$, then f has a continuous extension on $\overline{D}$, harmonic on D, given by $h_f(x) = \int_C P(z,x)\, f(z)\, \lambda(dz)$, where λ denotes normalized Lebesgue measure on C.

Not every harmonic function on D is obtained from a continuous function. By the Riesz–Herglotz theorem, however, there is for each $h \in \mathcal{H}^1$ a probability measure (all measures are Borel) ν_h on C such that

$$h = \int_C P(z,\cdot)\, \nu_h(dz).$$

The mapping $z \to P(z,\cdot)$ from C into $\mathcal{H}^1$ is a homeomorphism. We may think of ν_h as either a measure on C or on the collection of harmonic functions $\{P(z,\cdot): z \in C\}$. The latter point of view is that of Martin boundary and Choquet theories. The simplest realization of Choquet theory deals with a triangle. Each point inside and on a triangle is represented by a unique affine weight on the extreme points of the triangle, i.e., on the vertices. The set $\mathcal{H}^1$ is convex and compact in the ucc topology. The extreme points are the functions $\{P(z,\cdot): z \in C\}$. Each $h \in \mathcal{H}^1$ is represented by a unique probability measure ν_h on this set.

Fix $h \in \mathcal{H}^1$. As we shall see, the usual construction of ν_h is simple for the disk. This construction does not, however, generalize without going to an ideal boundary. We will give a construction that does generalize using results from [19], [20] and [23].

First, we recall that on the circle C_r, harmonic measure μ_x^r is the measure that gives the value at x of the harmonic extension of a continuous function on C_r. Moreover, normalized Lebesgue measure on C_r is the harmonic measure $\mu_{x_0}^r$. Given $h \in \mathcal{H}^1$, the measures $h \cdot \mu_{x_0}^r$ have weak–$*$ limit ν_h on C as $r \to 1$. Although this does not work in general, here is a method stated for the disk that does. First, for the disk, let $\{A_i^r\}$ form an interval partition of C_r, and choose $y_i^r \in A_i^r$. Let $\delta_{y_i^r}$ denote unit mass at the point y_i^r. The family of measures $\Sigma_i\, h(y_i^r) \cdot \mu_{x_0}^r(A_i^r) \cdot \delta_{y_i^r}$ also has the weak–$*$ limit ν_h on C as $r \to 1$ and the partition becomes finer and finer. This is because this finite sum of point masses is the measure that gives the Riemann sum approximation to the integral of a continuous function against the measure $h \cdot \mu_{x_0}^r$. We will move these point masses to a new space. Let δ_i^r be unit mass on the function equal to $\mu_x^r(A_i^r)/\mu_{x_0}^r(A_i^r)$ inside C_r and 0 on and outside C_r; this function is in the space $[0,+\infty]^D$ supplied with the product topology. The product topology is the ucc topology on the set $\{h > 0\colon h \text{ harmonic on } D_r,\ h(x_0) = 1\}$. Now, $\Sigma_i\, h(y_i^r) \cdot \mu_{x_0}^r(A_i^r) \cdot \delta_i^r$ has weak–$*$ limit ν_h on $\{P(z,\cdot)\colon z \in C\} \subset \mathcal{H}^1 \subset [0,+\infty]^D$ as $r \to 1$ and the partition becomes finer. This result continues to be true for quite general potential theories. (See [20] and [23].) It is the only potential theoretic general construction that I know that does not use the Martin boundary.

The nonstandard proof of this result in [20] used a construction from [19] that was the first use of the general measure theory after coin tossing and the first use of the standard part map to move a measure. The proof starts with a circle C_r with $r < 1$ but $r \simeq 1$. After suppressing the superscript r, we have for each $x \in D$,

$$h(x) = \int_{C_r} {}^*h(y)\, d\mu_y \simeq \sum_i {}^*h(y_i)\, \mu_{x_0}(A_i)[\mu_x(A_i)/\mu_{x_0}(A_i)].$$

The family of weights ${}^*h(y_i)\, \mu_{x_0}(A_i)$ is made into an ordinary probability measure (using the general theory) on the set of nonstandard harmonic functions $\mu_x(A_i)/\mu_{x_0}(A_i)$ and then projected to a representing measure on $\mathcal{X}^1$ via the standard part map. The process preserves affine combinations, so the final measure is the unique one on the extreme harmonic functions (Corollary of a result by Cartier, Fell, and Meyer, see [19].)

From this work on representing measures has come other work using nonstandard analysis in potential theory. The main thrust is a study of a Martin–type boundary for general domains. See [19], [22], and [6].

VII. FURTHER APPLICATIONS

Here is a partial list of some other applications of the general theory:

1) H. J. Keisler's work establishing a new strong existence theorem for stochastic differential equations in [16].

2) N. J. Cutland's papers on control theory in [7] and [8].

3) E. A. Perkins work (awarded the Rollo Davidson Prize in Probability Theory) on problems in the theory of local time ([28], [29], [30].) One striking result strengthened a classical theorem of Levy by combining exceptional sets of measure 0 depending on a space variable x into a single exceptional null set working uniformly in x.

4) T. Kamae's paper on the ergodic theorem in [15].

5) D. Hoover, E. Perkins and T. Lindstrom's work on stochastic integration in [12] and [17].

6) The large body of work in mathematical economics by R. M. Anderson, D. J. Brown, A. Khan, H. J. Keisler, C. Lewis, and S. Rashid.

7) Papers on infinite particle systems and thermodynamic limits by A. E. Hurd [13], L. L. Helms and P. A. Loeb [10], [11], together with new results in the book by Albeverio et al. [1].

8) L. Arkeryd solved a 100 year old problem by obtaining a Solution of the Boltzmann equation corresponding to specified periodic boundary conditions and quite general L^1 initial conditions in [4] and [5].

9) There are recent papers on descriptive set theory by C. W. Henson and D. Ross, and by H. J. Keisler, K. Kunen, A. Miller, and S. Leth.

10) There are four new books on the subject, [1], [14], [32] and the collection of papers [9].

References

1. S. Albeverio, J. E. Fenstad, R. Hoegh–Krohn and T. Lindstrom, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press Series on Pure and Applied Mathematics 122, Orlando, 1986..

2. R. M. Anderson, A nonstandard representation for Brownian motion and Itô integration, *Israel J. Math.* 25(1976), 15–46.

3. R. M. Anderson and S. Rashid. A nonstandard characterization of weak convergence, *Proc. Amer. Math. Soc.* 69(1978), 327–332.

4. L. Arkeryd, Loeb solutions of the Boltzmann equation, *Arch. Rational Mech. Anal.* 86(1984), 85–97.

5. ________, On the Boltzmann equation in unbounded space far from equilibrium, and the limit of zero mean free path, Commun. Math. Phys. 105(1986), 205–219.

6. J. Bliedtner and P. A. Loeb, An alternative to Martin boundary theory, *Semesterbericht Functionalanalysis*, Summer Semester 1988, Tübingen University, 55–60.

7. N. J. Cutland, Internal controls and relaxed controls, *J. London Math.* Soc. 27

8. ________, Optimal controls for partially observed stochastic systems: an infinitesimal approach. *Stochastics* 8.

9. N. J. Cutland editor. *Nonstandard Analysis and its Applications*, London Math. Soc. Student Texts #10, Cambridge Press, Cambridge, 1988,

10. L. L. Helms and P. A. Loeb, Applications of nonstandard analysis to spin models, *J. Math. Anal. Appl.* 69(1979) no. 2, 341–352.

11. ________, Bounds on the oscillation of spin systems, *J. Math. Anal. Appl.* 86(1982), 493–502.

12. D. H. Hoover and E. A. Perkins, Nonstandard construction of the stochastic integral and applications to stochastic differential equations I and II, *Trans. Amer. Math. Soc.* 275(1983), 1–36 and 37–58.

13. A. E. Hurd, Nonstandard analysis and lattice statistical mechanics: a variational principle, *Trans. Amer. Math. Soc.* 263(1981), 89–110.

14. A. E. Hurd and P. A. Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press Series on Pure and Applied Mathematics, 1985.

15. T. Kamae, A simple proof of the ergodic theorem, *Israel J. Math.* 42(1982), 284–290.

16. H. J. Keisler, An infinitesimal approach to stochastic analysis, *Mem. Amer. Math. Soc*, 48(1984), no 297.

17. T. L. Lindstrom, Hyperfinite stochastic integration I, II and III, *Math. Scand.* 46(1980), 265–292, 293–314 and 315–331.

18. P. A. Loeb, Conversion from nonstandard to standard measure spaces and applications in probability theory, *Trans. Amer. Math. Soc.* 211(1975), 113–122.

19. ________, Applications of nonstandard analysis to ideal boundaries in potential theory, *Israel J. Math.* 25(1976), 154–187.

20. _________, A generalization of the Riesz–Herglotz Theorem on representing measures, *Proc. Amer. Math. Soc.* 71(1978) no. 1, 65–68.

21. _________, Weak limits of measures and the standard part map, *Proc. Amer. Math. Soc.*, 77(1979) no. 1, 128–135.

22. _________, A regular metrizable boundary for solutions of elliptic and parabolic differential equations, *Math. Annalen* 251(1980), 43–50.

23. _________, A construction of representing measures for elliptic and parabolic differential equations, *Math. Annalen* 260(1982), 51–56.

24. _________, A functional approach to nonstandard measure theory, in Proceedings of the S. Kakutani retirement conference, *Contemporary Mathematics*, Vol. 26(1984), 251–261.

25. _________, Measure spaces in nonstandard models underlying standard stochastic processes, *Proceedings of 1983 International Congress of Mathematicians in Warsaw*, 323–335, PWN, Warsaw, 1984.

26. _________, A nonstandard functional approach to Fubini's Theorem, *Proc. Amer. Math. Soc.*, Vol. 93(1985) no. 2, 343–346.

27. _________, A lattice formulation of real and vector valued integrals, *Nonstandard Analysis and its Applications*, edited by Nigel Cutland, London Math. Soc. Student Texts #10, Cambridge Press, Cambridge, 1988, 221–236.

28. E. A. Perkins, A global intrinsic characterization of Brownian local time, *Ann. Probability* 9(1981), 800–817.

29. _________, The exact Hausdorff measure of the level sets of Brownian motion, *Z. Wahrscheinlichkeitstheorie verw. Geb.* 58(1981), 373–388.

30. _________, Weak invariance principles for local time, *Z. Wahrscheinlichkeitstheorie verw. Geb.* 60(1982), 437–451.

31. A Robinson, *Non–standard Analysis*, North–Holland, Amsterdam, 1966.

32. K. D. Stroyan and J. M. Bayod, *Foundations of Infinitesimal Stochastic Analysis*, North–Holland Studies in Logic Vol 119, Amsterdam, 1986.