数理解析研究所講究録
第 704 巻 1989 年 1-6

Solvability of convolution equations in Gevrey classes of Roumieu type

Rüdiger W. Braun, Reinhold Meise, Dietmar Vogt
(Düsseldorf Univ.)

Let $\mathcal{E}^{\{d\}}(\mathbb{R}^N)$ denote the Gevrey class of exponent $d > 1$ of Roumieu type. Cattabriga [2] has shown that in general not every constant coefficient linear partial differential equation $P(D)f = g$ has a solution $f \in \mathcal{E}^{\{d\}}(\mathbb{R}^N)$ for arbitrary $g \in \mathcal{E}^{\{d\}}(\mathbb{R}^N)$. In [9], Zampieri gave sufficient conditions on $P(D)$ for the solvability of the above equation in $\mathcal{E}^{\{d\}}(\mathbb{R}^N)$. He used ideas of Hörmander [4], who has characterized those $P(D)$ which act surjectively on the real-analytic functions on $\mathbb{R}^N$, i.e. on $\mathcal{E}^{\{1\}}(\mathbb{R}^N)$.

Recently, we have been able to characterize those convolution operators T_μ on the ultradifferentiable functions $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ of Roumieu type which are surjective. In the present note we state this result for the Gevrey classes $\mathcal{E}^{\{d\}}(\mathbb{R})$ and give a rough sketch of its proof. For details we refer to our forthcoming paper [1].

<u>1. Definition.</u> For $d > 1$ we define the Gevrey class $\mathcal{E}^{\{d\}}(\mathbb{R})$ of Roumieu type and exponent d by

$$\mathcal{E}^{\{d\}}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) \mid \text{for each compact set } K \subset \mathbb{R} \text{ there exists } h > 0: \sup_{x \in K} \sup_{p \in \mathbb{N}_0} \frac{|f^{(p)}(x)|}{h^p (p!)^d} < \infty\}.$$

We endow $\mathcal{E}^{\{d\}}(\mathbb{R})$ with its usual topology, which is obtained by taking first the inductive limit over $h > 0$ and then the projective limit of all compact sets K in $\mathbb{R}$ (see Komatsu [5],2.5). Furthermore we define

$$\mathcal{D}^{\{d\}}(\mathbb{R}) := \operatorname*{ind}_{n \to} \mathcal{D}^{\{d\}}[-n,n],$$

where

$$\mathcal{D}^{\{d\}}[-n,n] := \{f \in \mathcal{E}^{\{d\}}(\mathbb{R}) \mid \operatorname{Supp}(f) \subset [-n,n]\},$$

endowed with the topology induced by $\mathcal{E}^{\{d\}}(\mathbb{R})$.

2. Convolution operators

For $d > 1$ and $\mu \in \mathcal{E}^{\{d\}}(\mathbb{R})'$ we define the convolution operator T_μ on $\mathcal{E}^{\{d\}}(\mathbb{R})$, which is induced by μ, in the following way

$$T_\mu(f) : x \mapsto \langle \mu_y, f(x-y) \rangle, \; f \in \mathcal{E}^{\{d\}}(\mathbb{R}).$$

It is easy to check that T_μ is a continuous linear endomorphism of $\mathcal{E}^{\{d\}}(\mathbb{R})$.

For $\nu \in \mathcal{D}^{\{d\}}(\mathbb{R})'$ we define $\mu * \nu$ by

$$\langle \mu*\nu, \varphi \rangle := \langle \nu, \check{\mu}*\varphi \rangle, \; \varphi \in \mathcal{D}^{\{d\}}(\mathbb{R}),$$

where $\langle \check{\mu}, g \rangle := \langle \mu_x, g(-x) \rangle$ for $g \in \mathcal{E}^{\{d\}}(\mathbb{R})$. It is easy to see that $\mu*\nu$ is in $\mathcal{D}^{\{d\}}(\mathbb{R})'$.

Furthermore we define the Fourier-Laplace transform $\hat{\mu}$ of μ by

$$\hat{\mu} : \mathbb{C} \to \mathbb{C}, \; \hat{\mu}(z) := \langle \mu_x, e^{-ixz} \rangle,$$

and we put

$$V(\hat{\mu}) := \{z \in \mathbb{C} \mid \hat{\mu}(z) = 0\}.$$

Using the notation introduced above, we can state our main result:

3. Theorem. For $d > 1$ and $\mu \in \mathcal{E}^{\{d\}}(\mathbb{R})'$, the following conditions are equivalent:

(1) $T_\mu : \mathcal{E}^{\{d\}}(\mathbb{R}) \to \mathcal{E}^{\{d\}}(\mathbb{R})$ is surjective.

(2) μ satisfies (i) and (ii):

- (i) there exists $\nu \in \mathcal{D}^{\{d\}}(\mathbb{R})'$ with $\mu*\nu = \delta$, i.e. T_μ admits a fundamental solution
- (ii) $V(\hat{\mu})$ can be decomposed as $V(\hat{\mu}) = V_0 \,\dot{\cup}\, V_1$ with

$$\lim_{\substack{|z|\to\infty \\ z\in V_0}} \frac{|\mathrm{Im}\, z|}{|z|^{1/d}} = 0 \quad \text{and} \quad \liminf_{\substack{|z|\to\infty \\ z\in V_1}} \frac{|\mathrm{Im}\, z|}{|z|^{1/d}} > 0.$$

4. Remark. (a) In Theorem 3, condition (2)(ii) can be replaced by each of the following conditions (iii) or (iv)

(iii) ker T_μ is reflexive (or bornological, or quasi-barrelled)

(iv) ker T_μ is linear topologically isomorphic to $X \times Y$ for some nuclear Fréchet space X and some (DFN)-space Y.

(b) In Theorem 3.(2)(ii) the term $|z|^{1/d}$ can be replaced by $|\mathrm{Re}\, z|^{1/d}$.

To sketch the proof of Theorem 3, fix $d > 1$ and $n \in \mathbb{N}$. Then put

$$\mathcal{E}^{\{d\}}(n) := \{f \in \mathcal{E}^{\{d\}}(\mathbb{R}) \mid f(x) = 0 \text{ for all } x \in [-n,n]\}$$

and define $\mathcal{E}_n := \mathcal{E}^{\{d\}}(\mathbb{R})/\mathcal{E}^{\{d\}}(n)$.

Next fix $\mu \in \mathcal{E}^{\{d\}}(\mathbb{R})'$ and choose $m \in \mathbb{N}$ with $\mathrm{Supp}(\mu) \subset [-m,m]$. Then it is easy to check that T_μ induces for each $n \in \mathbb{N}$ a continuous linear map

$$\tau_n : \mathcal{E}_{n+m} \to \mathcal{E}_n \text{ by } \tau_n(f+\mathcal{E}^{\{d\}}(n+m)) := T_\mu(f)+\mathcal{E}^{\{d\}}(n).$$

Let $\mathcal{K}(\mu,d)$ denote the projective spectrum consisting of the spaces $(\ker \tau_n)_{n\in\mathbb{N}}$ and the natural restriction maps. Using properties of the projective limit functor of Palamodov [7], one can see that the following holds:

5. Lemma. For $d > 1$ let $\mu \in \mathcal{E}^{\{d\}}(\mathbb{R})'$ be given. Assume that there exists $\nu \in \mathcal{D}^{\{d\}}(\mathbb{R})'$ with $\mu * \nu = \delta$. Then T_μ is surjective on $\mathcal{E}^{\{d\}}(\mathbb{R})$ if and only if $\mathrm{Proj}^1 \mathcal{K}(\mu,d) = 0$.

To evaluate the condition $\mathrm{Proj}^1 \mathcal{K}(\mu,d) = 0$ we use a recent result of Vogt [8], by which $\mathrm{Proj}^1 \Lambda(A) = 0$ is characterized for certain countable projective spectra $\Lambda(A)$ of (DF)-sequence spaces. To state his result, we introduce the following notation.

<u>6. Definition.</u> Assume that $A = (a_{j,k,m})_{j,k,m\in\mathbb{N}}$ satisfies the following conditions for all $j,k,m \in \mathbb{N}$:

(1) $a_{j,k,m} > 0$ (2) $a_{j,k,m} \leq a_{j,k+1,m}$ (3) $a_{j,k,m} \geq a_{j,k,m+1}$.

Then we fix $k,m \in \mathbb{N}$ and define

$$\lambda(k,m) := \{x \in \mathbb{C}^{\mathbb{N}} \mid \|x\|_{k,m} := \sum_{j=1}^{\infty} |x_j| a_{j,k,m} < \infty\}.$$

Then $(\lambda(k,m), \|\ \|_{k,m})$ is a Banach space and because of (3) the inclusion map $j^k_{m,m+1} : \lambda(k,m) \to \lambda(k,m+1)$ is continuous. Hence we can define

$$\lambda(k) := \operatorname*{ind}_{m\to\infty} \lambda(k,m).$$

Then (2) implies that the inclusion $\iota^k_{k+1} : \lambda(k+1) \to \lambda(k)$ is continuous. By $\Lambda(A)$ we denote the projective spectrum $(\lambda(k), \iota^k_{k+1})_{k\in\mathbb{N}}$.

<u>7. Example.</u> Let $\alpha = (\alpha_j)_{j\in\mathbb{N}}$ and $\beta = (\beta_j)_{j\in\mathbb{N}}$ be sequences of non-negative numbers with $\lim_{j\to\infty} \beta_j = \infty$. Then

$$A_{\alpha,\beta} := (\exp(k\alpha_j + \tfrac{1}{m}\beta_j))_{j,k,m\in\mathbb{N}}$$

obviously satisfies the conditions (1)-(3) in Definition 6. In this case, $\lambda(k)$ is a (DFS)-space for each $k \in \mathbb{N}$ and we shall denote $\operatorname*{proj}_{\leftarrow k} \lambda(k)$ by $\lambda(\alpha,\beta)$.

The result of Vogt [8], mentioned above, can now be stated as follows:

<u>8. Theorem</u> (Vogt [8]). Let $A = (a_{j,k,m})_{j,k,m\in\mathbb{N}}$ satisfy the conditions in 6. and let $\Lambda(A)$ denote the projective spectrum defined in 6. Then the following conditions are equivalent:

(1) $\mathrm{Proj}^1 \Lambda(A) = 0$

(2) $\forall \mu \in \mathbb{N}\ \exists n,k \in \mathbb{N}\ \forall m,L \in \mathbb{N}\ \exists N,S \in \mathbb{N}\ \forall j \in \mathbb{N}$:

$$\frac{1}{a_{j,k,m}} \le S \max\left(\frac{1}{a_{j,L,N}} , \frac{1}{a_{j,\mu,n}} \right).$$

The application of Theorem 8 is possible because of the following result:

<u>9. Theorem</u> (Meise [6]). For $d > 1$ let $\mu \in \mathcal{E}^{\{d\}}(\mathbb{R})'$ be given. Assume there exists $\nu \in \mathcal{D}^{\{d\}}(\mathbb{R})'$ with $\mu * \nu = \delta$ and $\dim \ker T_\mu = \infty$. Then $\ker T_\mu$ is linear topologically isomorphic to $\lambda(\alpha,\beta)$, where $\alpha = (|\operatorname{Im} a_j|)_{j\in\mathbb{N}}$, $\beta = (|a_j|^{1/d})_{j\in\mathbb{N}}$ for a sequence $(a_j)_{j\in\mathbb{N}}$ which counts the zeros of $\hat{\mu}$ with multiplicities.

<u>Sketch of proof of Theorem 3:</u> Arguments of Ehrenpreis [3] can be used to show that the surjectivity of T_μ implies the existence of a fundamental solution for T_μ. For the rest of the proof we can assume w.l.o.g. that $\dim \ker T_\mu = \infty$. Then Theorem 9 can be used to show that the projective spectra $\Lambda(A_{\alpha,\beta})$ and $K(\mu,d)$ are equivalent. By the properties of the projective limit functor, this implies $\operatorname{Proj}^1 K(\mu,d) = \operatorname{Proj}^1 \Lambda(A_{\alpha,\beta})$. Hence the result follows from Lemma 5 by evaluating condition 8.(2) for $A_{\alpha,\beta}$.

<u>10. Example.</u> There exists an ultradifferential operator $\mu \in \bigcap_{d>1} \mathcal{E}^{\{d\}}(\mathbb{R})'$ with the following properties:

(1) $V(\hat{\mu}) = \{\pm e^j \pm i e^k \mid j,k \in \mathbb{N}\}$

(2) $T_\mu : \mathcal{E}^{\{d\}}(\mathbb{R}) \to \mathcal{E}^{\{d\}}(\mathbb{R})$ admits a fundamental solution for each $d > 1$

(3) for each $d > 1$, $T_\mu : \mathcal{E}^{\{d\}}(\mathbb{R}) \to \mathcal{E}^{\{d\}}(\mathbb{R})$ is not surjective.

However, for each $d > 1$, $T_\mu : \mathcal{E}^{(d)}(\mathbb{R}) \to \mathcal{E}^{(d)}(\mathbb{R})$ is surjective, where $\mathcal{E}^{(d)}(\mathbb{R})$ denotes the Gevrey class of Beurling type of exponent d.

References

[1] Braun, R. W., Meise, R., Vogt, D.: Existence of fundamental solutions and surjectivity of convolution operators on classes of ultradifferentiable functions, preprint

[2] Cattabriga, L.: On the surjectivity of differential polynomials on Gevrey spaces, Atti del Convegno "Linear partial and pseudodifferential operators", 30 Sett. - 2 Otto. 1982

[3] Ehrenpreis, L.: Solutions of some problems of division IV, Amer. J. Math. 82, 522-588 (1960)

[4] Hörmander, L.: On the existence of real analytic solutions of partial differential equations with constant coefficients, Invent. Math. 21, 151-183 (1973)

[5] Komatsu, H.: Ultradistributions I: Structure theorems and a characterization, J. Fac. Sci. Univ. Tokyo Sect. IA, 24, 607-628 (1973)

[6] Meise, R.: Sequence space representations for zero-solutions of convolution equations on ultradifferentiable functions of Roumieu type, preprint

[7] Palamodov, V. P.: The projective functor in the category of linear topological spaces, Math. USSR-Sbornik 4, 529-559 (1968)

[8] Vogt, D.: Projective spectra of (DF)-spaces, preprint

[9] Zampieri, G.: An application of the fundamental principle of Ehrenpreis to the existence of global Gevrey solutions of linear differential equations, Boll. U.M.I. (6), 5-B, 361-392 (1986)