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Rademacher Series and Self-affine functions

Yasunobu Shiota and Takeshi Sekiguchi
(塩田安信) (関口健)

Department of Mathematics, Akita University
Tegata, Akita 010, Japan
and
College of General Education, Tôhoku Gakuin University
Ichinazaka, Sendai 981-31, Japan

0. Introduction.

Let $R_n(x)$ $(n=1,2,\cdots)$ be the nth Rademacher function, that is, $R_n(x) = 1 - 2\varepsilon_n(x)$, where $\varepsilon_n(x)$ is the nth digit of the (finite) binary expansion of $x \in [0,1)$. In this note we will deal with the the Rademacher series

$$f_r(x) = \sum_{n=1}^{\infty} r^n R_n(x) \quad (0 < r < 1).$$

1) The distribution of f_r.

Let f be a real function defined on $[0,1]$. We will consider f as a random variable on the probability space $([0,1),dx)$ and define its probability distribution by

$$\mu_f(E) = |\{x\in[0,1);\ f(x)\in E\}| \quad \text{for} \quad E \in \mathscr{B}(R),$$

where $|\cdot|$ denotes the one-dimensional Lebesgue measure. If μ_f is absolutely continuous with respect to the Lebesgue measure, we denote the density by α_f.

Theorem (Jessen and Wintner (1935)). The distribution μ_{f_r} is either absolutely continuous or singular with respect to the Lebesgue measure.

It is well-known that μ_{f_r} is singular for $0 < r < 1/2$ and that μ_{f_r} is absolutely continuous for $r = 2^{-1/d}$ with a positive integer d,

Theorem (Erdös (1939)). Let $1/2 < r < 1$. If r^{-1} is a Pisot number, that is, an algebraic number whose other conjugates lie inside the unit circle, then μ_{f_r} is singular.

Theorem (Salem (1943)). Let $1/2 < r < 1$. Then r^{-1} is a Pisot number if and only if the Fourier-Stieltjes transform $\hat{\mu}_{f_r}(\xi)$ of μ_{f_r} does not tend to zero as $|\xi|\to\infty$.

It is unknown for which values of r the distribution μ_{f_r} is absolutely continuous. Garsia (1962) gave several sufficient conditions for μ_{f_r} to be absolutely continuous or to be singular.

2) **Hausdorff dimension of level sets of** f_r.

The following result is referred in Mandelbrot's (1982) book.

Theorem (Beyer (1962)). If $r = 2^{-1/d}$ with a positive integer d, then the Hausdorff dimension of the level set of f_r is equal to $1-(1/d)$.

3) Fat baker's transformation.

Alexander and Yorke (1984) introduced the notion of the fat baker's transformation and point out its connection to our Rademacher series.

1. Self-affine functions and Dimensions.

Definition (Kôno (1986)). Let m be a positive integer > 1. A real function g defined on [0,1] is said to be a self-affine function with the scale parameter $0 < H \leq 1$ to base m if the relation

$$g((j+x)m^{-N}) - g(jm^{-N}) = T_{N,j}m^{-NH}\{g(x) - g(0)\}$$

holds for any jm^{-N} $(j=0,1,\cdots,m^N-1,\ N=1,2,\cdots)$ and $x \in [0,1)$, where

$\tau_{N,j} = 1$ or -1.

We denote the graph of a function f by $G(f)$, the Hausdorff dimension of a set E by $\dim_H(E)$ and the packing dimension (Taylor and Tricot (1985)) of a set E by $\dim_P(E)$. We note that $\dim_H(E) \leqq \dim_P(E)$ for any set E.

Theorem (Kôno (1986)). _Let_ g _be a self-affine function with the scale parameter_ $0 < H \leqq 1$. _Assume that the distribution_ μ_g _is absolutely continuous with respect to the Lebesgue measure and that the density_ α_g _belongs to_ $L^p(-\infty,\infty)$ _for every_ $1 < p < \infty$. _Then we have_ $\dim_H(G(g)) \geqq 2-H$.

Theorem. _Let_ g _be a bounded self-affine function with the scale parameter_ $0 < H \leqq 1$. _Then we have_ $\dim_P(G(g)) \leqq 2-H$.

Proposition. _The function_ f_r $(0 < r < 1)$ _is a bounded self-affine function with the scale parameter_ $H = \log_2(1/r)$ _to base_ 2. _Here we set_ $f_r(1) = f_r(1-)$.

Theorem. _Let_ $1/2 < r < 1$. _Assume that the distribution_ μ_{f_r} _is absolutely continuous with respect to the Lebesgue measure and that the density_ α_{f_r} _belongs to_ $L^p(-\infty,\infty)$ _for any_ $1 < p < \infty$. _Then we have_ $\dim_H(G(f_r)) = \dim_P(G(f_r)) = 2-\log_2(1/r)$.

Corollary. _If_ $r = 2^{-1/d}$ _with a positive integer_ d, _then we_ have $\dim_H(G(f_r)) = \dim_P(G(f_r)) = 2-(1/d)$.

Remark. 1) Przytycki and Urbański proved that if μ_{f_r} is absolutely continuous, then we have $\dim_H(G(f_r)) = 2-\log_2(1/r)$. We note that their result requires no assumption on the density α_{f_r}. Indeed, it is remarked in their paper that the absolute continuity of μ_{f_r} implies the boundedness of α_{f_r}. Furthermore, by the use of Erdös's result, they obtained the following: If r^{-1} is a Pisot number, then $1 < \dim_H(G(f_r)) < 2-\log_2(1/r)$.
2) For a continuous self-affine function, Kôno (1988) gave a necessary and sufficient condition for the distribution to be absolutely continuous and then, by the use of Kôno's result, Urbański obtained the exact formula of the Hausdorff dimension of the graph of a self-affine function. On the other hand, Bertoin computed the Hausdorff dimension of the level set of a continuous self-affine function.

2. Self-affine functions and Functional equations.

Theorem (Hutchinson (1981)). Let (X,d) be a complete metric space and $\{f_j\}_{1\leq j\leq m}$, $2 \leq m < \infty$, be a set of the contractions on X.
1) There is a unique compact subset $K = K(f_1,f_2,\cdots,f_m)$ of X such that the equality $K = \bigcup_{j=1}^{m} f_j(K)$ holds.
2) For any compact subset E of X, we have $\lim_{n\to\infty} F^n(E) = K$ in the

Hausdorff metric, where F is defined by $F(E) = \bigcup_{j=1}^{m} f_j(E)$.

In the following we suppose that a function G may take two values $G(jm^{-N})$ and $G(jm^{-N}-)$ at $x = jm^{-N}$ $(N=1,2,\cdots)$.

Theorem (de Rham (1957), Hata (1984)). Suppose that $\{f_j\}_{1\leqq j\leqq m}$ is the same as in the above theorem.

1) The functional equation

$$G(x) = \begin{cases} f_1(G(mx)) & \text{for } 0 \leqq x \leqq \frac{1}{m}, \\ \quad\vdots & \\ f_m(G(mx-(m-1))) & \text{for } \frac{m-1}{m} \leqq x \leqq 1, \end{cases}$$

has a unique continuous solution $G: [0,1] \longrightarrow X$ if and only if

$$f_{j+1}(\mathrm{Fix}(f_1)) = f_j(\mathrm{Fix}(f_m))$$

holds for every $1 \leqq j \leqq m-1$. In this case the graph $G([0,1])$ is compact.

2) Suppose that the above functional equation has a solution $G: [0,1] \longrightarrow X$, whose range $G([0,1])$ is compact in X. Then we have $G([0,1]) = K(f_1,f_2,\cdots f_m)$. Furthermore, if X is a vector space in addition, we can approximate $G([0,1])$ by broken-lines: Define

$$G^{(0)}(x) = \{\mathrm{Fix}(f_m)-\mathrm{Fix}(f_1)\}x + \mathrm{Fix}(f_1)$$

and, for $n = 1,2,\cdots,$

$$G^{(n)}(x) = \begin{cases} f_1(G^{(n-1)}(mx)) & \text{for } 0 \leqq x \leqq \frac{1}{m}, \\ \vdots & \\ f_m(G^{(n-1)}(mx-(m-1))) & \text{for } \frac{m-1}{m} \leqq x \leqq 1. \end{cases}$$

Then we have $K(f_1,f_2,\cdots,f_m) = \lim_{n\to\infty} G^{(n)}([0,1])$ *in the Hausdorff metric*.

We now consider a characterization of self-affine function by a functional equation. We suppose that a self-affine function g satisfies the equality

$$g((j+1)m^{-N}-) - g(jm^{-N}) = T_{N,j}m^{-NH}\{g(1-) - g(0)\}$$

for any jm^{-N} $(j=0,1,\cdots,m^N-1,\ N=1,2,\cdots)$.

Theorem. *A real function* g *defined on* $[0,1]$ *with* $g(0) = 0$ *is a self-affine function with the scale parameter* $0 < H \leqq 1$ *to base* m *if and only if it satisfies the following functional equations*:

$$g(x) = \begin{cases} a_1 m^{-H} g(mx) & \text{for } 0 \leqq x \leqq \frac{1}{m}, \\ a_2 m^{-H} g(mx-1) + b_2 & \text{for } \frac{1}{m} \leqq x \leqq \frac{2}{m}, \\ \vdots & \\ a_m m^{-H} g(mx-(m-1)) + b_m & \text{for } \frac{m-1}{m} \leqq x \leqq 1, \end{cases}$$

where $a_j = 1$ or -1 *and* b_j *is a costant for each* j.

It is also possible to treat the self-affine function as a curve in the plane.

Corollary. *Define*

$$f_j : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1/m & 0 \\ 0 & a_j m^{-H} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} j-1/m \\ b_j \end{pmatrix}$$

for $j = 1,2,\cdots,m$, *where* $a_j = 1$ *or* -1 *and* b_j *is a constant for each* j *except for* $b_0 = 0$. *Then* $g(x)$ *is a self-affine function with the scale parameter* $0 < H \leqq 1$ *to base* m *if and only if* $G(x) = (x,g(x))$ *satisfies the functional equation*

$$G(x) = \begin{cases} f_1(G(mx)) & \text{for} \quad 0 \leqq x \leqq \frac{1}{m}, \\ \quad\vdots & \\ f_m(G(mx-(m-1))) & \text{for} \quad \frac{m-1}{m} \leqq x \leqq 1. \end{cases}$$

If $G([0,1])$, *which is the graph of* g, *is compact, then it coincides with the unique compact subset* $K = K(f_1,f_2,\cdots,f_m)$ *satisfying* $K = \bigcup_{j=1}^{m} f_j(K)$.

We normalize f_r by setting

$$g_r(x) = (1/2) - ((1-r)/2r)f_r(x) = ((1-r)/r)\sum_{n=1}^{\infty} r^n \varepsilon_n(x).$$

Then g_r is also a self-affine function with the scale parameter $H = \log_2(1/r)$ to base 2 and satisfies the following functional equation:

$$g_r(x) = \begin{cases} rg_r(2x) & \text{for } 0 \leqq x \leqq \frac{1}{2}, \\ rg_r(2x-1) + (1-r) & \text{for } \frac{1}{2} \leqq x \leqq 1. \end{cases}$$

If we define

$$f_1: \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1/2 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$f_2: \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1/2 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1-r \end{pmatrix},$$

then $G_r(x) = (x, g_r(x))$ satisfies the functional equation

$$G_r(x) = \begin{cases} f_1(G_r(2x)) & \text{for } 0 \leqq x \leqq \frac{1}{2}, \\ f_2(G_r(2x-1)) & \text{for } \frac{1}{2} \leqq x \leqq 1. \end{cases}$$

Since f_1 and f_2 are contractions, there is a unique compact subset $K = K(f_1, f_2)$ of R^2 satisfying $K = f_1(K) \cup f_2(K)$. In this case, $f_1(\mathrm{Fix}(f_2)) \neq f_2(\mathrm{Fix}(f_1))$ and hence the functional equation has no continuous solution $G_r: [0,1] \longrightarrow R^2$. However, it is easily seen that $G_r([0,1])$ is compact. Hence we have $K = G_r([0,1])$ and $\lim_{n\to\infty} G_r^{(n)}([0,1]) = K$ in the Hausdorff metric.

We next consider the relation between h_r and Lebesgue's singular function. Define

$$\bar{f}_1 : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1/2 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\bar{f}_2 : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 1/2 & 0 \\ 0 & 1-r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ r \end{pmatrix}.$$

Since $\bar{f}_1$ and $\bar{f}_2$ are contractions, there is a unique compact subset $\bar{K} = \bar{K}(f_1, f_2)$ of R^2 satisfying $\bar{K} = \bar{f}_1(\bar{K}) \cup \bar{f}_2(\bar{K})$. In this case, since $\bar{f}_2(\mathrm{Fix}(\bar{f}_1)) = \bar{f}_1(\mathrm{Fix}(\bar{f}_2))$, the functional equation

$$\bar{G}_r(x) = \begin{cases} \bar{f}_1(\bar{G}_r(2x)) & \text{for } 0 \leqq x \leqq \frac{1}{2}, \\ \bar{f}_2(\bar{G}_r(2x-1)) & \text{for } \frac{1}{2} \leqq x \leqq 1 \end{cases}$$

has a unique continuous solution $\bar{G}_r : [0,1] \longrightarrow R^2$ (de Rham (1957), Hata and Yamaguti (1984)). If we set $\bar{G}_r(x) = (x, \bar{g}_r(x))$, the above functional equation is equivalent to

$$\bar{g}_r(x) = \begin{cases} r\bar{g}_r(2x) & \text{for } 0 \leqq x \leqq \frac{1}{2}, \\ (1-r)\bar{g}_r(2x-1) + r & \text{for } \frac{1}{2} \leqq x \leqq 1. \end{cases}$$

The function $\bar{g}_r$ is so-called Lebesgue's singular function. Since $\bar{G}_r([0,1])$ is compact, $\lim_{n\to\infty} \bar{G}_r^{(n)}([0,1]) = K$ in the Hausdorff metric. This is nothing but a Salem's (1943) geometric construction of Lebesgue's singular function.

We finally remark that the distribution function $F_r(x) = \mu_{g_r}((-\infty,x])$ satisfies the functional equation

$$F_r(x) = \frac{1}{2}\{F_r(\frac{x}{r}) + F_r(\frac{x}{r} - \frac{1-r}{r})\};$$

$$F_r(x) = 0 \quad \text{for} \quad x \leqq 0; \quad F_r(x) = 1 \quad \text{for} \quad x \geqq 1.$$

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