数理解析研究所講究録
第 707 巻 1989 年 1-11

An Inner Product Inequality Which Appears in Analytic Number Theory*

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ABSTRACT. Selberg's inequality which has its orign in the analytic theory of numbers will be discussed. The authors developed *diagonal majorization method* to prove Selberg's inequality. There are some applications of Selberg's inequality and of the *method* itself.

* AMS(MOS) Subject Classification (1984) : 46C05, 15A45, 11N35 65F15.

Key Words and Phrases : Selberg's Inequality, Geršgorin's Theorem.

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§1. Introduction.

There is a lot of variants and generalizations of the well-known Cauchy - Bunyakovskiĭ- Schwarz' inequality:

THEOREM CBS. *If x, y are vectors in an inner product space $\mathcal{H}$, then*

$$|<x \mid y>|^2 \leq \|x\|^2\|y\|^2.$$

In Bombieri's text [3] on analytic number theory, a variant of the Cauchy - Bunyakovskiĭ- Schwarz' inequality is referred to A. Selberg. The inequality goes as follows:

THEOREM S. *If $x_1, x_2, \ldots, x_n$, and x are non zero vectors in an inner product space $\mathcal{H}$, then*

$$\sum_{i=1}^{n} \frac{|<x \mid x_i>|^2}{\sum_{j=1}^{n}|<x_i \mid x_j>|} \leq \|x\|^2.$$

It is easy to see that this inequality is nothing but the Cauchy-Bunyakovskiĭ-Schwarz' inequality if $n = 2$, and Bessel's inequality

$$\sum_{i=1}^{n}|<x \mid x_i>|^2 \leq \|x\|^2$$

if the vectors $x_1, x_2, \ldots, x_n$ are chosen to form an orthonormal system.

In Bombieri's text [3], a proof to Selberg's inequality which is similar to the well-known proof of the Cauchy-Bunyakovskiĭ-Schwarz' inequality is given.

In the first part of this talk, we obtained another proof to the Selberg's inequality, based on what we call *diagonal majorization method* hereafter. By this method, we mean a general algorithm to obtain a diagonal majorant of a given positive semidefinite matrix. Several inequalities have matrices whose positive semidefiniteness is equivalent to the inequality itself. While R. Bellman [1] emphasized the importance of the identity that makes an inequality trivial, we call attention to the importance of getting positive semidefinite matrix that makes an inequality trivial. From this viewpoint, there can be some inequalities related to Selberg's.

In the second part, we will talk about a few examples of applications of Selberg's. The examples are chosen from the theory of positive semidefinite functions on semigroups.

At last, we will talk about another application of the *diagonal majorization method*.

This is a note of my talk at the Research Institute of Mathematical Sciences, Kyoto University. The detailed paper will be published elsewhere.

§2. A Proof of Selberg's Inequality.

It is quite attractive that the Cauchy - Bunyakovskiĭ- Schwarz' inequality or Bessel's inequality is equivalent to the positive semidefiniteness of the following 2×2 or $(n+1) \times (n+1)$ matrix respectively:

$$\begin{pmatrix} \|x\|^2 & < x \mid y > \\ < y \mid x > & \|y\|^2 \end{pmatrix},$$

or

$$\begin{pmatrix} \|x\|^2 & < x \mid x_1 > & \dots & < x \mid x_n > \\ < x_1 \mid x > & 1 & & O \\ \vdots & & \ddots & \\ < x_n \mid x > & O & & 1 \end{pmatrix},$$

respectively.

Thus we think it natural to ask some matrix whose positive semidefiniteness is equivalent to Selberg's inequality. A candidate for such matrix is given as follows:

$$S := \begin{pmatrix} \|x\|^2 & < x \mid x_1 > & \ldots & < x \mid x_n > \\ < x_1 \mid x > & \sum_{j=1}^n \lvert < x_1 \mid x_j > \rvert & & O \\ \vdots & & \ddots & \\ < x_n \mid x > & O & & \sum_{j=1}^n \lvert < x_n \mid x_j > \rvert \end{pmatrix}.$$

Thus we have only to show the positive semidefiniteness of this $(n+1) \times (n+1)$ matrix. And we call attention to the fact that these matrices are offsprings of the so-called Gram's matrix. The definition of Gram's matrix goes as follows:

DEFINITION. Let $x_1, x_2, \ldots, x_n$ be an n-ple of vectors in an inner product space $\mathcal{H}$. The Gram matrix of the x_i's denoted by $\mathbf{G}(x_1, x_2, \ldots, x_n)$ is given by the following equation:

$$\mathbf{G}(x_1, x_2, \ldots, x_n) = \begin{pmatrix} < x_1 \mid x_1 > & < x_1 \mid x_2 > & \ldots & < x_1 \mid x_n > \\ < x_2 \mid x_1 > & < x_2 \mid x_2 > & \ldots & < x_2 \mid x_n > \\ \vdots & \vdots & \ddots & \vdots \\ < x_n \mid x_1 > & < x_n \mid x_2 > & \ldots & < x_n \mid x_n > \end{pmatrix}.$$

It is well-known that a Gram matrix of arbitrary size is positive semidefinite. Remark also that the Gram matrix $\mathbf{G}(x, x_1, \ldots, x_n)$ differs from the matrix S in the $n \times n$ lower right square. Comparing them, you will be suggested the following majorization theorem, which makes the positivity of the desired matrix trivial.

LEMMA. *If $x_1, x_2, \ldots, x_n$, and x are vectors in an inner product space $\mathcal{H}$, then*

$$\mathbf{G}(x_1, x_2, \ldots, x_n) \leq \mathbf{diag}\Big(\sum_{j=1}^n \lvert < x_1 \mid x_j > \rvert, \sum_{j=1}^n \lvert < x_2 \mid x_j > \rvert, \ldots, \sum_{j=1}^n \lvert < x_n \mid x_j > \rvert\Big),$$

where

$$\mathbf{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \begin{pmatrix} \alpha_1 & & & O \\ & \alpha_2 & & \\ & & \ddots & \\ O & & & \alpha_n \end{pmatrix}$$

We have a proof of this LEMMA using the well-known eigenvalue-location theorem due to Geršgorin. Of course there can be a proof without using the eigenvalue location theorem, but we have a clear perspective from the location theorem.

THEOREM G (Geršgorin, cf., [5]). *Let $A = [a_{ij}] \in M_n(\mathbf{C})$, and let*

$$R_i(A) = \sum_{j \neq i, j=1}^{n} |a_{ij}|, \quad 1 \leq i \leq n$$

denote the deleted absolute row sums of A. Then all the eigenvalues of A are located in the union of n discs, (so called Geršgorin discs)

$$\bigcup_{i=1}^{n} \{z \in \mathbf{C} : |z - a_{ii}| \leq R_i(A)\} := G(A).$$

PROOF OF LEMMA. Consider the following matrix

$$\mathbf{diag}\Big(\sum_{j=1}^{n} |< x_1 \mid x_j >|, \sum_{j=1}^{n} |< x_2 \mid x_j >|, \ldots, \sum_{j=1}^{n} |< x_n \mid x_j >|\Big)$$
$$-\mathbf{G}(x_1, x_2, \ldots, x_n)$$
$$= \begin{pmatrix} \sum_{j\neq 1} |< x_1 \mid x_j >| & -< x_1 \mid x_2 > & \ldots & -< x_1 \mid x_n > \\ -< x_2 \mid x_1 > & \sum_{j\neq 2} |< x_2 \mid x_j >| & \ldots & -< x_2 \mid x_n > \\ \vdots & \vdots & \ddots & \vdots \\ -< x_n \mid x_1 > & -< x_n \mid x_2 > & \ldots & \sum_{j\neq n} |< x_n \mid x_j >| \end{pmatrix},$$

whose Geršgorin discs obviously lie in the right half plane, and hence the eigenvalues lie in the right half of the real axis. Thus the matrix is positive semidefinite.

Q.E.D.

Beckenbach and Bellman shows a refinement of the Cauchy - Bunyakovskiĭ-Schwarz' inequality in their text [1]. The *diagonal majorization method* is available to prove the refinement, but much more. It will be shown that the *method* implies the following refinement of Selberg's inequality. The proof will be omitted here.

COROLLARY. *If $x_1, x_2, \ldots, x_n$, and x, y are non zero vectors in an inner product* *space, then*

$$| < x \mid y > - \sum_{i=1}^{n} \frac{< x \mid x_i >< x_i \mid y >}{\sum_{j=1}^{n} | < x_i \mid x_j > |} |^2$$
$$\leq$$
$$(\|x\|^2 - \sum_{i=1}^{n} \frac{| < x \mid x_i > |^2}{\sum_{j=1}^{n} | < x_i \mid x_j > |})(\|y\|^2 - \sum_{i=1}^{n} \frac{| < y \mid x_i > |^2}{\sum_{j=1}^{n} | < x_i \mid x_j > |}).$$

§3. Applications of Selbergs Inequality.

Through the representation theorem of the positive definite functions, cf., [2], Selberg's inequality yields several inequalities. Only a few results will be introduced here.

PROPOSITION 1.

$$\sum_{i=1}^{n} \frac{\cos^2(x - x_i)}{\sum_{j=1}^{n} |\cos(x_i - x_j)|} \leq 1,$$

for $x, x_1, x_2, \ldots, x_n \in \mathbf{R}$ *satisfying*

$$x_i - x_j \neq \frac{(2N+1)\pi}{2}, \quad (N \notin \mathbf{Z}, 1 \leq i, j \leq n).$$

Set $n = 2$ in PROPOSITION 1, one obtains the following inequality:

COROLLARY.

$$|\cos x_1 \mp \cos x_2| \leq |\sin x_1 \pm \sin x_2|,$$

for $x_1, x_2 \in \mathbf{R}$ *satisfying*

$$\cos(x_1 - x_2) \geq_{<} 0.$$

In the branch of probability theory, we obtain the following.

PROPOSITION 2.

$$\sum_{i=1}^{n} \frac{(P(A \cap A_i) - P(A)P(A_i))^2}{\sum_{j=1}^{n} (P(A_i \cap A_j) - P(A_i)P(A_j))} \leq P(A)(1 - P(A)),$$

for $A \in \mathbf{A}$ *and pairwise independent* $A_1, A_2, \ldots, A_n \in \mathbf{A}$, *where* $(\Omega, \mathbf{A}, P)$ *denotes a probability space.*

A Cauchy-Bunyakovskiĭ-Schwarz' inequality with a linear operator weight was discussed by T. Furuta [4].

THEOREM F. *For any bounded linear operator* T *on a Hilbert space* $\mathcal{H}$, *vectors* $x, y \in \mathcal{H}$, *and any real number* $\alpha \in (0,1)$, *the inequality*

$$|< Tx \mid y >|^2 \leq < |T|^{2\alpha} x \mid x >< |T|^{2(1-\alpha)} y \mid y >$$

holds true.

Let $T = U|T|$ be the polar decomposition. With a couple of replacements of vectors

$$x \mapsto |T|^\alpha x, \quad \text{and} \quad x_i \mapsto |T|^{1-\alpha} U^* x_i,$$

in Selberg's inequality gives the following weighted form of Selberg's inequality.

COROLLARY. *Let* T *be a bounded linear operator on a Hilbert space* $\mathcal{H}$, *and* $\alpha \in (0,1)$. *If* $x_1, x_2, \ldots, x_n \notin \operatorname{Ker}(T)$, *and* x *are vectors in* $\mathcal{H}$, *then*

$$\sum_{i=1}^{n} \frac{|< Tx \mid x_i >|^2}{\sum_{j=1}^{n} |< |T^*|^{2(1-\alpha)} x_i \mid x_j >|} \leq \||T|^\alpha x\|^2.$$

Set $n = 1$, and we have THEOREM F. Of coures, we have the refinement of this COROLLARY in the same way as that of Selberg's.

§4. Another Application of Diagonal Majorization.

From the Euclidean or unitary world of Hilbert space, we shall immigrate ourselves into the hyperbolic world. Conside the unit disc $\mathcal{H}_1$ of the Hilbert space $\mathcal{H}$. Then the inner product $< x \mid y >$ in the Euclidean world $\mathcal{H}$ corresponds to the quantity

$$\frac{1}{1 - < x \mid y >}, \quad (x, y \in \mathcal{H}_1),$$

in the hyperbolic world $\mathcal{H}_1$. Thus we have the following matrix that corresponds to the Gram matrix.

DEFINITION. Let $x_1, x_2, \ldots, x_n$ be an n-ple of vectors in the open unit disc $\mathcal{H}_1$ of an inner product space $\mathcal{H}$. The Hua matrix of the x_i's denoted by $\mathbf{H}(x_1, x_2, \ldots, x_n)$ is given by the following equation:

$$H = \left[\frac{1}{1- < x_i \mid x_j >}\right]_{ij=1}^{n}$$

The following theorem corresponds to the positivity of Gram matrices.

THEOREM 1. *If $x_1, x_2, \ldots, x_n$ is an n-ple of vectors in the open unit disc of the inner product space H, then*

$$\mathbf{H}(x_1, x_2, \ldots, x_n)$$

is positive semidefinite.

PROOF. Since the inner products $< x_i \mid y_j >$ have modulus strictly less than 1, one can represent the entries $\frac{1}{1-<x_i \mid y_j>}$ as the power series:

$$\frac{1}{1- < x_i \mid y_j >} = \sum_{n=0}^{\infty} < x_i \mid y_j >^n .$$

And hence one has

$$\mathbf{H}(x_1, x_2, \ldots, x_n) = \sum_{n=0}^{\infty} \mathbf{G}(x_1, x_2, \ldots, x_n)^{(n)},$$

where $M^{(n)}$ denotes the power with respect to Schur (i.e., elementwise) product of a matrix M. It is trivial that the matrix

$$\mathbf{G}(x_1, x_2, \ldots, x_n)^{(0)} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \ldots \\ 1 & 1 & \ldots & 1 \end{pmatrix}$$

is positive semidefinite and is well known that the Gram matrix

$$\mathbf{G}(x_1, x_2, \ldots, x_n)$$

is positive semidefinite. Hence so are the powers with respect to Schur product. Thus the Hua matrix represented as the (Schur-) power series of Gram matrix is positive semidefinite.

Q.E.D.

We have shown that the positive semidefiniteness of Gram matrix yields not only the Cauchy-Bunyakovskiĭ-Schwarz inequality but also an inequality due to A. Selberg. Just in the same way, we obtain a hyperbolic analogy of Selberg's inequality from the positive semidefiniteness of Hua matrix. Before describing the statement, the concept of parallel sum must be introduced.

DEFINITION. Let $a_1, a_2, \ldots, a_n$ be an n-ple of positive real numbers. Then their *parallel sum* is defined by

$$(\sum_{i=1}^{n} a_i^{-1})^{-1},$$

and is denoted by

$$\prod_{i=1}^{n} : a_i.$$

THEOREM 2. *If $x_1, x_2, \ldots, x_n$, and x are vectors in the open unit disc $\mathcal{H}_1$ of an inner product space $\mathcal{H}$, then*

$$1 - \|x\|^2 \leq \prod_{i=1}^{n} : \frac{|1- < x \mid x_i > |^2}{\prod_{j=1}^{n} : |1- < x_i \mid x_j > |}.$$

The proof will be omitted.

The following refinement of the preceding inequality is obtained in the same way.

COROLLARY. *If $x_1, x_2, \ldots, x_n$, and x, y are vectors in the open unit disc of an* *inner product space, then*

$$|\prod_{i=1}^{n}:\frac{(1-<x\mid x_i>)(1-<x_i\mid y>)}{\prod_{j=1}^{n}:|1-<x_i\mid x_j>|}-1+<x\mid y>|^2$$
$$\leq$$
$$(\prod_{i=1}^{n}:\frac{|1-<x\mid x_i>|^2}{\prod_{j=1}^{n}:|1-<x_i\mid x_j>|}-1+\|x\|^2)$$
$$\times$$
$$(\prod_{i=1}^{n}:\frac{|1-<y\mid x_i>|^2}{\prod_{j=1}^{n}:|1-<x_i\mid x_j>|}-1+\|y\|^2).$$

As an example for the application, the following inequality is given.

PROPOSITION 2'.

$$\prod_{i=1}^{n}:\frac{(1-P(A\cap A_i)+P(A)P(A_i))^2}{\prod_{j=1}^{n}:(1-P(A_i\cap A_j)+P(A_i)P(A_j))}\geq 1-P(A)(1-P(A)),$$

for $A\in\mathbf{A}$ and pairwise independent $A_1,A_2,\ldots,A_n\in\mathbf{A}$, where $(\Omega,\mathbf{A},P)$ denotes a probability space.

It is well known that the unit disk is conformally equivalent to the upper (or right) half plane. Hence it is natural to ask for the conformal equaivalent of the inequality in THEOREM 2 for the complex plane.

The first step is to prove the following positive semidefiniteness of the matrix corresponding to Hua matrix.

THEOREM 3. *If $z_1,z_2,\ldots,z_n$ is an n-ple of complex numbers. then the $n\times n$ matrix*

$$M=M(z_1,z_2,\ldots,z_n)$$

defined by

$$M(z_1,z_2,\ldots,z_n)=\left[\frac{1}{z_i+\overline{z_j}}\right]_{ij=1}^{n}$$

is positive semidefinite.

As a consequence of the preceding theorem and the *diagonal majorization method,* we have the following inequality.

THEOREM 4. *If $z, z_1, z_2, \ldots, z_n$ is an $n+1$-ple of complex numbers in the open upper half plane $\Gamma = \{z \in \mathbf{C} : \Re(z) \geq 0\}$. Then*

$$z + \overline{z} \leq \prod{}_{i=1}^{n} \frac{|z + \overline{z_i}|^2}{\prod{}_{j=1}^{n} |z_i + \overline{z_j}|}.$$

In concluding my talk, we would like to express my hearty thanks to Prof. T. Ando for many valuable suggestions for further study.

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ADDED IN PROOF. The results stated here are first introduced by Prof. F. Kubo at the Eleventh Symposium on Applied Functional Analysis. Hearing these results, Prof. T. Furuta has realized me the interest of the equality condition for Selberg's inequality. (*T. Furuta, When does the equality of Selberg type extension of Heinz inequality hold ?, Preprint.*) Thus Prof. M. Fujii, the organizer of the present symposium urged him to give another talk. He also send me a note on an elementary proof of the LEMMA of the *diagonal majorization method.*