数理解析研究所講究録
第 710 巻 1989 年 116-126

Bifurcation Equations of 3-Dimensional Piecewise-Linear Vector Fields

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ABSTRACT

Global bifurcation equations of 3-dimensional 3-region continuous piecewise-linear vector fields with odd symmetry are given.
The bifurcation equations cover all sort of bifurcation sets, i.e., homoclinicity, heteroclinicity, saddle-node bifurcation, period-doubling bifurcation and Hopf bifurcation.

1. Normal forms for 3-dimensional 3-region systems.

Consider a 3-dimensional 3-region continuous piecewise linear vector field $f:\mathbb{R}^3 \dashrightarrow \mathbb{R}^3$ with odd symmetry $f(-x)=-f(x)$. Assume f has no proper linear subspace which is invariant under linear vector field in the middle region, and which is parallel to the boundary. If f has eigenvalues μ_1,μ_2,μ_3 at the middle region, and ν_1,ν_2,ν_3 at the outer regions, the normal form theorem for piecewise linear vector fields guarantees that such a vector field is uniquely determined up to the linearly conjugacy as follows;

$$f(x) = Ax + \frac{1}{2} p \{|\langle\alpha,x\rangle-1| + (\langle\alpha,x\rangle-1)\}$$

$$- \frac{1}{2} p \{|\langle\alpha,x\rangle+1| - (\langle\alpha,x\rangle+1)\}$$

$$= \begin{cases} B(x-P) & (x \in R_+) \\ Ax & (x \in R_0) \\ B(x+P) & (x \in R_-) \end{cases} \tag{1.1}$$

where

$R_\pm = \{x \in \mathbb{R}^3 : \pm\langle\alpha,x\rangle-1 > 0 \}$,

$R_0 = \{x \in R^3 : |\langle\alpha,x\rangle| \le 1 \}$,

$\alpha = {}^T(1,0,0)$, $p = {}^T(c_1,c_2,c_3)$,

$P = {}^T(1-a_3/b_3, c_1a_3/b_3, c_2a_3/b_3)$,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3 & a_2 & a_1 \end{pmatrix}, \ B = \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ c_3+a_3 & a_2 & a_1 \end{pmatrix} = A + p^T\alpha$$

$a_1=\mu_1+\mu_2+\mu_3$, $a_2=-(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)$, $a_3= \mu_1\mu_2\mu_3$,

$b_1=\nu_1+\nu_2+\nu_3$, $b_2=-(\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1)$, $b_3= \nu_1\nu_2\nu_3$,

$c_1=b_1-a_1$, $c_2= b_2-a_2+c_1a_1$, $c_3=b_3-a_3+a_2c_1+a_1c_2$.

Define the boundaris U_1 and U_{-1} by

$U_{\pm1} = \{x \in \mathbb{R}^3: \langle\alpha,x\rangle = \pm1 \}$.

We separate U_1 and U_{-1} into two parts respectively;

$U_1^+ = \{x \in U_1: \langle\alpha,Ax\rangle > 0\}$, $U_1^- = \{x \in U_1: \langle\alpha,Ax\rangle < 0\}$,

$U_{-1}^+ = \{x \in U_1: \langle-\alpha,Ax\rangle > 0\}$, $U_{-1}^- = \{x \in U_1: \langle-\alpha,Ax\rangle < 0\}$.

Define points $C_i \in U_1$ (i=1,2,3) by

$$C_i = {}^T(1,\mu_i,\mu_i^2) , \tag{1.2}$$

then the vector $\overrightarrow{OC_i}$ is a eigenvector of A with respect to μ_i (i=1,2,3). Define points $D_i \in U_1$ (i=1,2,3) by

$$D_i = {}^T(1, \nu_i a_3/b_3, \nu_i(\nu_i - c_1)a_3/b_3), \tag{1.3}$$

then the vector $\overrightarrow{PD_i}$ is a eigenvector of B with respect to ν_i (i=1,2,3).

2. Return time coordinete.

Assume the matricies A and B are reguler.

(1)For $x \in U_1^+$, assume that there are $y,z \in U_1^-$ such that

$$y = e^{-At}x \qquad \text{where } t = \inf\{t'>0 : |\langle e_1, e^{-At'}x\rangle| = 1\}$$

$$z = e^{Bs}(x-P)+P \text{ where } s = \inf\{s'>0 : \langle e_1, e^{Bs'}(x-P)+P\rangle = 1\}.$$

Since $Aw = B(w-P)$ for all $w \in U_1$ by continuity of the vector field we have

$$z = A^{-1}A(e^{Bs}(x-P)+P) = A^{-1}Be^{Bs}(x-P) = A^{-1}e^{Bs}B(x-P)$$

$$= A^{-1}e^{Bs}Ax = e^{Cs}x$$

where $C = A^{-1}BA$. Since x,y and z belong to U_1,

$${}^T\alpha e^{-At}x = 1,\ {}^T\alpha x = 1,\ {}^T\alpha e^{Cs}x = 1,$$

or equivarentry

$$[e_1{}^T\alpha e^{-At} + e_2{}^T\alpha + e_3{}^T\alpha e^{Cs}]\ x = h_1$$

where $e_1 = {}^T(1,0,0)$, $e_2 = {}^T(0,1,0)$, $e_3 = {}^T(0,0,1)$, $h_1 = {}^T(1,1,1)$.

If the matrix [...] is reguler, we denote the inverse matrix by

$$K(t,s) = [e_1{}^T\alpha e^{-At} + e_2{}^T\alpha + e_3{}^T\alpha e^{Cs}]^{-1}. \tag{2.1}$$

Then we have

$$x = K(t,s)h_1. \tag{2.2}$$

The (t,s) is called the *return time coordineite* of x.

(2)For $x \in U_1^+$, assume that there are $y \in U_{-1}^-$ and $z \in U_1^-$ such that

$$y = e^{-At}x \qquad \text{where } t = \inf\{t'>0 : |\langle e_1, e^{-At'}x\rangle| = 1\}$$

$$z = e^{Bs}(x-P)+P \text{ where } s = \inf\{s'>0 : \langle e_1, e^{Bs'}(x-P)+P\rangle = 1\}.$$

Since $Aw = B(w-P)$ for all $w \in U_1$ by continuity of the vector field, we have

$$z = A^{-1}A(e^{Bs}(x-P)+P) = A^{-1}Be^{Bs}(x-P) = A^{-1}e^{Bs}B(x-P)$$

$$= A^{-1}e^{Bs}Ax = e^{Cs}x$$

where $C = A^{-1}BA$. Since $x,z \in U_1$ and $y \in U_{-1}$,

$${}^T\alpha e^{-At}x = -1,\ {}^T\alpha x = 1,\ {}^T\alpha e^{Cs}x = 1,$$

or equivarentry

$$[e_1{}^T\alpha e^{-At}+e_2{}^T\alpha+e_3{}^T\alpha e^{Cs}]\ x = h_2 \tag{2.3}$$

where $e_1={}^T(1,0,0)$, $e_2={}^T(0,1,0)$, $e_3={}^T(0,0,1)$, $h_2={}^T(-1,1,1)$.

If the matrix [...] is reguler, we denote the inverse matrix by

$$K(t,s) = [e_1{}^T\alpha e^{-At}+e_2{}^T\alpha+e_3{}^T\alpha e^{Cs}]^{-1}.$$

Then we have

$$x = K(t,s)h_2. \tag{2.4}$$

The (t,s) is also called the *return time coordineite* of x.

3. Periodic orbit.

(1)The type of $U_1 \longrightarrow U_1$.

Assume $x \in U_1^+$ has a return time coordinate (t_1,s_1) as follows;

$x = K(t_1,s_1)h_1$ where

$$t_1 = \inf\{t > 0 : |\langle e_1, e^{-At}x\rangle| = 1\} \quad \text{and} \tag{3.1a}$$

$$s_1 = \inf\{s > 0 : \langle e_1, e^{Bs}(x-P)+P\rangle = 1\}. \tag{3.1b}$$

If an equation between 6 eigenvalues $\mu_1,\mu_2,\mu_3,\nu_1,\nu_2,\nu_3$ and (t_1,s_1) defined by

$$e^{-At_1}K(t_1,s_1) = e^{Cs_1}K(t_1,s_1) \tag{3.2}$$

holds under the open condition (3.1), the point $x = K(t_1,s_1)$ is a periodic point for a vector field determined by $\mu_1,\mu_2,\mu_3,\nu_1,\nu_2,\nu_3$. This periodic point is called of the type $U_1 \longrightarrow U_1$. For the other type of periodic points, the equations are derived by similar way.

(2)The type of $U_1 \to U_1 \to U_1 \to U_1$.

$$e^{Cs_1}K(t_1,s_1)h_1 = e^{-At_2}K(t_2,s_2)h_1, \tag{3.3a}$$

$$e^{Cs_2}K(t_2,s_2)h_1 = e^{-At_1}K(t_1,s_1)h_1. \tag{3.3b}$$

The open conditions for the first return times;

$|{}^T\alpha e^{-At}K(t_i,s_i)h_1| \neq 1$ for all t with $0 < t < t_i$,

${}^T\alpha e^{Cs}K(t_i,s_i)h_1 \neq 1$ for all s with $0 < s < s_i$ $(i = 1,2)$.

(3)The type of $U_1 \to U_1 \to U_{-1} \to U_{-1} \to U_1$.

$$e^{Cs_1}K(t_1,s_1)h_2 = -e^{-At_2}K(t_2,s_2)h_2 , \tag{3.4a}$$

$$-e^{Cs_2}K(t_2,s_2)h_2 = e^{-At_1}K(t_1,s_1)h_2 . \tag{3.4b}$$

The open conditions for the first return times;

$|{}^T\alpha e^{-At}K(t_i,s_i)h_2| \neq 1$ for all t with $0 < t < t_i$,

${}^T\alpha e^{Cs}K(t_i,s_i)h_2 \neq 1$ for all s with $0 < s < s_i$ $(i = 1,2)$.

(4)The type of $U_1 \to U_1 \to U_1 \to U_1 \to U_1 \to U_{-1} \to U_{-1} \to U_1$.

$$e^{Cs_1}K(t_1,s_1)h_2 = e^{-At_2}K(t_2,s_2)h_1 , \tag{3.5a}$$

$$e^{Cs_2}K(t_2,s_2)h_1 = -e^{-At_3}K(t_3,s_3)h_2 , \tag{3.5b}$$

$$-e^{Cs_3}K(t_3,s_3)h_2 = e^{-At_1}K(t_1,s_1)h_2 . \tag{3.5c}$$

The open conditions for first return times;

$$|{}^{T}\alpha e^{-At}K(t_i,s_i)h_j| \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_i,$$

$${}^{T}\alpha e^{Cs}K(t_i,s_i)h_j \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_i \quad (i = 1,2),$$

where $j = 1$ $(i = 2)$; $= 2$ $(i = 1,3)$.

4. Bifurcation conditions of periodic orbits.

Let $x = K(t_1,s_1)h_1$ be a periodic point with a period $t_1+s_1+\ldots+t_n+s_n$ as stated in § 3. Then bifurcation conditions for this periodic point are given as follows. Set

$$M = e^{Bs_n}e^{At_n}e^{Bs_{n-1}}e^{At_{n-1}}\ldots e^{Bs_1}e^{At_1},$$

$$T = \mathrm{Trace}(M) \quad \text{and} \quad D = \det(M).$$

(1) *Saddle-node bifurcation;*

$$2 - T + D = 0. \tag{4.1}$$

(2) *Period doubling bifurcation;*

$$T + D = 0. \tag{4.2}$$

(3) *Hopf bifurcation;*

$$D - 1 = 0 \quad \text{and} \quad (1-T)^2 - 4 < 0. \tag{4.3}$$

5. Homoclinic orbits at O.

Suppose μ_1 is positive real, and μ_2 and μ_3 are negative reals or complex conjugate pair with negative real part. Since an eigen vector for μ_i is given by (1.2), a 2-dimensional stable manifold $W^s(\mathbf{0})$ and a 1-dimensional unstable manifold $W^u(\mathbf{0})$ for $\mathbf{0} = {}^{T}(0,0,0)$ are given as follows;

$$W^u(\mathbf{0}) = \{\, x \in \mathbb{R}^3 : x = r\overrightarrow{\mathbf{OC}}_1,\ {}^{T}\alpha x - 1 \leq 0,\ r \in \mathbb{R} \,\},$$

$$W^s(\mathbf{O}) = \{ x \in \mathbb{R}^3 : x = r\overrightarrow{\mathbf{OC}}_2 + r'\overrightarrow{\mathbf{OC}}_3,\ {}^T\alpha x - 1 \le 0,\ r, r' \in \mathbb{R} \}.$$

Set

$$u = {}^T(0,\ (\mu_2+\mu_3)/(\mu_2\mu_3),\ -1/(\mu_2\mu_3)),$$

then the intersection $W^s(\mathbf{O}) \cap U_1$ is given by

$$W^s(\mathbf{O}) \cap U_1 = \{x=(x,y,z) \in \mathbb{R}^3 : {}^Tux-1 = 0,\ x=1\}.$$

Then a condition under which a homoclinic orbit at $\mathbf{O}$ exists is given as follows.

(1)The type of $\mathbf{O} \to U_1 \to U_1 \to \mathbf{O}$.

$$ {}^T\alpha e^{Cs_0}\mathbf{C}_1 = 1, \tag{5.1a}$$

$$ {}^Tue^{Cs_0}\mathbf{C}_1 = 1. \tag{5.1b}$$

The open conditions for the first return times;

$${}^T\alpha e^{Cs}\mathbf{C}_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_0,$$

$${}^T\alpha e^{At}e^{Cs_0}\mathbf{C}_1 \neq 1 \text{ for all } t > 0.$$

(2)The type of $\mathbf{O} \to U_1 \to U_1 \to U_1 \to U_1 \to \mathbf{O}$.

$$ {}^T\alpha e^{Cs_0}\mathbf{C}_1 = 1, \tag{5.2a}$$

$$ e^{Cs_0}\mathbf{C}_1 = e^{-At_1}K(t_1,s_1)\mathbf{h}_1, \tag{5.2b}$$

$$ {}^Tue^{Cs_1}K(t_1,s_1)\mathbf{h}_1 = 1. \tag{5.2c}$$

The open conditions for the first return times;

$${}^T\alpha e^{Cs}\mathbf{C}_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_0,$$

$$|{}^T\alpha e^{-At}K(t_1,s_1)\mathbf{h}_1| \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_1,$$

$${}^T\alpha e^{Cs}K(t_1,s_1)\mathbf{h}_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_1.$$

$$|{}^T\alpha e^{At}e^{Cs_1}K(t_1,s_1)\mathbf{h}_1| \neq 1 \quad \text{for all } t > 0.$$

(3)The type of $\mathbf{O} \to U_1 \to U_1 \to U_{-1} \to U_{-1} \to \mathbf{O}$.

$$^{T}\alpha e^{Cs_0}C_1 = 1 , \tag{5.3a}$$

$$e^{Cs_0}C_1 = -e^{-At_1}K(t_1,s_1)h_2 , \tag{5.3b}$$

$$^{T}ue^{Cs_1}K(t_1,s_1)h_2 = 1 . \tag{5.3c}$$

The open conditions for the first return times;

$$^{T}\alpha e^{Cs}C_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_0,$$

$$|^{T}\alpha e^{-At}K(t_1,s_1)h_2 | \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_1 ,$$

$$^{T}\alpha e^{Cs}K(t_1,s_1)h_2 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_1 .$$

$$|^{T}\alpha e^{At}e^{Cs_1}K(t_1,s_1)h_2| \neq 1 \quad \text{for all } t > 0.$$

6. Homoclinic orbits at P^+.

For eigenvalues of B, assume ν_1 is negative real, and ν_2 and ν_3 are positive reals or complex conjugate pair with positive real part. Since an eigen vector for ν_i is given by (1.3), a 1-dimensional stable manifold $W^s(P)$ and a 2-dimensional unstable manifold $W^u(P)$ for P are given as follows;

$$W^s(P) = \{ x \in \mathbb{R}^3: x = r\overrightarrow{PD}_1 + P, \ ^{T}\alpha x-1\geq 0, \ r\in\mathbb{R} \},$$

$$W^u(P)=\{x \in \mathbb{R}^3: x = r\overrightarrow{PD}_2 + r'\overrightarrow{PD}_3 + P, \ ^{T}\alpha x-1 \geq 0, \ r,r'\in\mathbb{R} \}.$$

Set

$$v = {}^{T}(0, (\nu_2+\nu_3-c_1)b_3/(\nu_2\nu_3a_3), -b_3/(\nu_2\nu_3a_3)),$$

then the intersection $W^u(P) \cap U_1$ is given by

$$W^u(P) \cap U_1 = \{x=(x,y,z)\in \mathbb{R}^3: \ ^{T}vx-1 = 0 , x=1\}.$$

Set

$$P^+ = P \quad \text{and} \quad P^- = -P.$$

Then a condition under which a homoclinic orbit at P^+ exists is given as follows.

(1)The type of $P^+ \to U_1 \to U_1 \to U_1 \to U_1 \to P^+$.

$$ {}^T\nu e^{-At_1} K(t_1,s_1)h_1 = 1, \tag{6.1a} $$

$$ e^{Cs_1} K(t_1,s_1)h_1 = e^{-At_2} D_1. \tag{6.1b} $$

The open conditions for the first return times;

$$ {}^T\alpha e^{Cs} e^{-At_1} K(t_1,s_1)h_1 \neq 1 \quad \text{for all } s > 0 , $$

$$ |{}^T\alpha e^{-At} K(t_1,s_1)h_1| \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_1, $$

$$ {}^T\alpha e^{Cs} K(t_1,s_1)h_1 \neq 1 \quad \text{for all } s \text{ with } 0 < s < s_1 , $$

$$ |{}^T\alpha e^{-At} D_1| \neq 1 \quad \text{for all } t \text{ with } 0 < t < t_2. $$

7. Heteroclinic orbits.

For eigenvalues of A, assume μ_1 is positive real, and μ_2 and μ_3 are negative reals or complex conjugate pair with negative real part. For eigenvalues of B, assume ν_1 is negative real, and ν_2 and ν_3 are positive reals or complex conjugate pair with positive real part. Then a condition under which a heteroclinic orbit from **0** or P^- to P^+ exists is given as follows.

(1)The type of $\mathbf{0} \to U_1 \to P^+$.

$$ C_1 = D_1. \tag{7.1} $$

(2)The type of $\mathbf{0} \to U_1 \to U_1 \to U_1 \to P^+$.

$$ {}^T\alpha e^{Cs_0} C_1 = 1 , \tag{7.2a} $$

$$ e^{Cs_0} C_1 = e^{-At_1} D_1 , \tag{7.2b} $$

$$^{T}\alpha e^{-At_1}D_1 = 1. \tag{7.2c}$$

The open conditions for the first return times;

$^{T}\alpha e^{Cs}C_1 \neq 1$ for all s with $0 < s < s_0$,

$|^{T}\alpha e^{-At}D_1| \neq 1$ for all t with $0 < t < t_1$.

(3)The type of $O \to U_1 \to U_1 \to U_1 \to U_1 \to U_1 \to P^+$.

$$^{T}\alpha e^{Cs_0}C_1 = 1 , \tag{7.3a}$$

$$e^{Cs_0}C_1 = e^{-At_1}K(t_1,s_1)h_1 , \tag{7.3b}$$

$$e^{Cs_1}K(t_1,s_1)h_1 = e^{-At_2}D_1 , \tag{7.3c}$$

$$^{T}\alpha e^{-At_2}D_1 = 1. \tag{7.3d}$$

The open conditions for the first return times;

$^{T}\alpha e^{Cs}C_1 \neq 1$ for all s with $0 < s < s_0$,

$|^{T}\alpha e^{-At}K(t_1,s_1)h_1| \neq 1$ for all t with $0 < t < t_1$,

$^{T}\alpha e^{Cs}K(t_1,s_1)h_1 \neq 1$ for all s with $0 < s < s_1$.

$|^{T}\alpha e^{-At}D_1| \neq 1$ for all t with $0 < t < t_2$.

(4)The type of $P^- \to U_{-1} \to U_1 \to P^+$.

$$^{T}\alpha e^{-At_0}C_1 = 1 , \tag{7.4a}$$

$$^{T}u e^{-At_0}C_1 = -1 . \tag{7.4b}$$

The open conditions for the first return times;

$|^{T}\alpha e^{-At}C_1| \neq 1$ for all t with $0 < t < t_0$.

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