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NORMAL HILBERT POLYNOMIALS

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1. Normal Hilbert Polynomials.

This note is a short summary of my recent work [5]. Throughtout this note (A, m) will be a Cohen-Macaulay local ring of dimension $d \geq 2$, and I will be a parameter ideal for A, i.e. I is an m-primary ideal generated by d elements. Assume that A is analytically unramified and A/m is infinite. For an ideal J in A, $\overline{J}$ denotes the integral closure of J, i.e., $\overline{J} = \{x \in A | x^n + a_1 x^{n-1} + \cdots + a_n = 0 \text{ for some } a_i \in J^i\}$.

It is well known that there exist uniquely determined integers $\overline{e}_0(I), \cdots, \overline{e}_d(I)$ such that

$$length_A(A/\overline{I^{n+1}}) = \overline{e}_0(I)\binom{n+d}{d} - \overline{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{e}_d(I)$$

for all large n. We say that

$$P(I, n) = \overline{e}_0(I)\binom{n+d}{d} - \overline{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{e}_d(I)$$

is the *normal Hilbert polynomial* of I. $\overline{e}_0(I)$ is a well-known number called the multiplicity of I i.e., $\overline{e}_0(I) = e(I) = length_A(A/I)$. Our purpose of this note is to report some properties of $\overline{e}_1(I)$, $\overline{e}_2(I)$ and $\overline{e}_3(I)$. Our results are contained in the following two theorems.

THEOREM 1. (1) $\overline{e}_1(I) - length_A(\overline{I}/I) \geq length_A(\overline{I^2}/I\overline{I})$, *and the equality holds if and only if* $\overline{I^{n+2}} = I^n \overline{I^2}$ *for every* $n \geq 0$.

(2) $\overline{e}_2(I) \geq \overline{e}_1(I) - length_A(\overline{I}/I)$, *and the equality holds if and only if* $\overline{I^{n+2}} = I^n \overline{I^2}$ *for every* $n \geq 0$.

THEOREM 2. (1) $\overline{e}_3(I) \geq 0$, *and if* $\overline{e}_3(I) = 0$, *then* $\overline{I^{n+2}}$ *is contained in* I^n *for every* $n \geq 0$.

(2) *Assume that A is Gorenstein and* $\overline{I} = m$. *Then* $\overline{e}_3(I) = 0$ *if and only if* $\overline{I^{n+2}} = I^n \overline{I^2}$ *for every* $n \geq 0$.

Ooishi called the number $\overline{e}_1(I) - length_A(\overline{I}/I)$ the *normal sectional genus* of I and denoted it by $\overline{g}_s(I)$.

In two dimensional case, Huneke remarked in his paper that

$$length_A(A/\overline{I^{n+1}}) = length_A(A/I)\binom{n+2}{2}$$
$$-(\sum_{r\geq 0} length_A(\overline{I^{r+1}}/I\overline{I^r}))\binom{n+1}{1} + \sum_{r\geq 1} length_A(\overline{I^{r+1}}/I\overline{I^r})r$$

for all large n ([3]). Therefore

$$\sum_{r\geq 1} length_A(\overline{I^{r+1}}/I\overline{I^r})r \geq \overline{g}_s(I) = \sum_{r\geq 1} length_A(\overline{I^{r+1}}/I\overline{I^r})$$
$$\geq length_A(\overline{I^2}/I\overline{I});$$

thus our Theorem 1 is a natural generalization of this fact to high dimensional case, and the main difficulty is how we reduce the problem to two dimensional case.

2. Key Theorem.

Detailed studies in $\overline{e}_i(I)$'s are based on the following theorem and lemma.

THEOREM 3. *There exists a system of generators $x_1, \cdots, x_d$ of I such that, if we put $C = A(T)/(\sum_i x_iT_i)$ and $J = IC$, where $A(T) = A[T]_{m[T]}$ with $T = (T_1, \cdots, T_d)d$ indeterminates, then*

(1) $\overline{J^n} \cap A = \overline{I^n}$ *for every* $n \geq 0$;

(2) $\overline{J} = \overline{I}C$;

(3) $\overline{J^n} = \overline{I^n}C \cong \overline{I^n}A(T)/(\sum_i x_iT_i)\overline{I^{n-1}}A(T)$ *for all large* n;

(4) *C is normal if A is analytically normal and* $dimA \geq 3$.

LEMMA 4. *Choose a system of minimal generators $x_1, \cdots, x_d$ of I, and put $B = A[x_1/x_2], R = R(A, I) = A[It, t^{-1}]$* (the Rees ring of I), *$D = R_P$, where $P = (t^{-1}, m)R$, and $nD =$* the maximal ideal of D. *Let $h : B \longrightarrow D$ be a canonical homomorphism which maps x_1/x_2 to x_1t/x_2t. Then*

(0) $nD \cap B = m[x_1/x_2]$.

We here put $C = Bm[x_1/x_2]$ and $J = IC$. Then

(1) $\overline{J^n} \cap A = \overline{I^n}$ *for every* $n \geq 0$,

(2) $\overline{J} = \overline{I}C$ *and*

(3) $\overline{I^{n+r}} \subseteq \overline{I^n}$ *for every* $n \geq 0$ *if and only if* $\overline{J^{n+r}} \subseteq \overline{J^n}$ *for every* $n \geq 0$.

Applying the theorem (and the lemma), we have the following results.

PROPOSITION 5. *With the same notation as in* Theorem 3, *we have the following assertions.*

(1) $\bar{e}_i(I) = \bar{e}_i(J)$ *for every* $i \leq d-1$;

(2) $\bar{g}_s(I) = \bar{g}_s(J)$;

(3) $\overline{I^2}A(T)/I\overline{I}A(T)$ *is a submodule of* $\overline{J^2}/J\overline{J}$, *in particular* $length_A(\overline{I^2}/I\overline{I}) \geq length_C(\overline{J^2}/I\overline{J})$.

(4) $\overline{I^{n+r}}$ *is contained in* I^n *for every* $n \geq 0$ *if and only if* $\overline{J^{n+r}}$ *is contained in* $\overline{J^n}$ *for every* $n \geq 0$.

PROOF. We put $z = \sum_i x_i T_i$ for simplicity. (1) and (2) follow from Theorem 3. (3): It is enough to show that $\overline{I^2}A(T) \cap (I\overline{I}, z)A(T) = I\overline{I}A(T)$. $\overline{I^2}A(T) \cap (I\overline{I}, z)A(T) = I\overline{I}A(T) + \overline{I^2 A(T)} \cap zA(T) = I\overline{I}A(T) + \overline{IA(T)}z = I\overline{I}A(T)$. (4) follows from Lemma 4(3).

It is known that $\bar{e}_2(I) \geq \bar{g}_s(I)$ if $dimA = 2$; therefore by the induction on $d = dimA$, we have

COROLLARY 6. $\bar{e}_2(I) \geq \bar{g}_s(I)$.

As proved in [4, Proposition 10],

$$
\begin{aligned}
(*) & length_A(A/\overline{I^{n+1}}) \\
& \geq length_A(A/I^{n+1}\overline{I^2}) \\
& = length_A(A/I)\binom{n+d}{d} - (length_A(\overline{I}/I) + length_A(\overline{I^2}/I\overline{I}))\binom{n+d-1}{d-1} \\
& \quad + length_A(\overline{I^2}/I\overline{I})\binom{n+d-2}{d-2}
\end{aligned}
$$

for all n. Therefore

$$
\begin{aligned}
& \bar{e}_1(I) \geq length_A(\overline{I}/I) + length_A(\overline{I^2}/I\overline{I}) i.e., \\
& \bar{g}_s(I) = \bar{e}_1(I) - length_A(\overline{I}/I) \geq length_A(\overline{I^2}/I\overline{I}).
\end{aligned}
$$

We give here the proof of (1) in Theorem 1.

PROPOSITION 7. (1) $\bar{g}_s(I) \geq length_A(\overline{I^2}/I\overline{I})$.

(2) $\overline{g}_s(I) = length_A(\overline{I^2}/I\overline{I})$ *if and only if* $\overline{I^{n+2}} = I^n\overline{I^2}$ *for every* $n \geq 0$.

PROOF. (2) We may assume that A/m is infinite. We use the induction on $d = dimA$. If $d = 2$, the assertion clearly holds. So assume that $d > 2$. if part follows from (*). only if part: Choose a system of generators $x_1, \cdots, x_d$ of I satisfying the conditions of Theorem 3, and put $z = \sum_i x_i T_i, C = A(T)/zA(T)$ and $J = IC$. Since $length_A(\overline{I^2}/I\overline{I}) \leq length_C(\overline{J^2}/J\overline{J}) \leq \overline{g}_s(J) = \overline{g}_s(I)$ by Proposition 5, we have $length_C(\overline{J^2}/J\overline{J}) = \overline{g}_s(J)$ and $\overline{J^2} = \overline{I^2}C$. Thus $\overline{J^{n+2}} = J^n\overline{J^2}$ for every $n \geq 0$. Then $\overline{I^{n+2}}A(T) = \overline{I^{n+2}A(T)}$ is contained in $I^n\overline{I^2}A(T) + zA(T)$, and hence $\overline{I^{n+2}}A(T) = \overline{I^{n+2}}A(T) \cap (I^n\overline{I^2}A(T) + zA(T)) = I^n\overline{I^2}A(T) + z\overline{I^{n+1}}A(T)$. By the induction on n, $\overline{I^{n+2}}A(T) = I^n\overline{I^2}A(T)$, and therefore $\overline{I^{n+2}} = I^n\overline{I^2}$.

As we remarked in [4, Proposition 3],

(**) if $\overline{I^{n+2}} = I^n\overline{I^2}$ for every $n \geq 0$, then $G = R'/t^{-1}R'$ is Cohen-Macaulay.

Since

$$length_A(A/\overline{I^{n+1}}) = length_A(A/I^{n+1}) - \sum_{0 \leq r \leq n} length_A(I^{n-r}\overline{I^{r+1}}/I^{n-r+1}\overline{I^r})$$

and

$$\begin{aligned}
&length_A(I^{n-r}\overline{I^{r+1}}/I^{n-r+1}\overline{I^r}) \\
&\leq length_A((\overline{I^{r+1}}/I\overline{I^r}) \otimes (I^{n-r}/I^{n-r+1})) \\
&= length_A(\overline{I^{r+1}}/I\overline{I^r})\binom{n-r+d-1}{d-1} \\
&= length_A(\overline{I^{r+1}}/I\overline{I^r})\binom{n+d-1}{d-1} - r\binom{n-r+d-2}{d-2} + \textit{lower degree terms},
\end{aligned}$$

we have

$$\overline{e}_1(I) \leq \sum_{r \geq 0} length_A(\overline{I^{r+1}}/I\overline{I^r}) \text{ i.e.,}$$

$$\overline{g}_s(I) = \overline{e}_1(I) - length_A(\overline{I}/I) \leq \sum_{r \geq 1} length_A(\overline{I^{r+1}}/I\overline{I^r}).$$

PROPOSITION 8. (1) $\overline{g}_s(I) \leq \sum_{r \geq 1} length_A(\overline{I^{r+1}}/I\overline{I^r})$.

(2) *If* $depth_M R' \geq d$, *then* $\overline{g}_s(I) = \sum_{r\geq 1} length_A(\overline{I^{r+1}}/I\overline{I^r})$ *and* $\overline{e}_2(I) = \sum_{r\geq 1} length_A(\overline{I^{r+1}}/I\overline{I^r})r$, *where* $M = (t^{-1}, It)R$.

3. $\overline{e}_2(I)$.

In this section, we shall give the proof of (2) in Theorem 1. By Corollary 6, the assertion remained to be proved is the following

PROPOSITION 9. $\overline{e}_2(I) = \overline{g}_s(I)$ *if and only if* $\overline{I^{n+2}} = I^n\overline{I^2}$ *for every* $n \geq 0$.

If part of the above proposition clearly holds by (*). *Only if* part follows from the following proposition.

PROPOSITION 10. *Assume that* $d \geq 3$, *and choose a system of generators* $x_1, \cdots, x_d$ *satisfying the conditions of* Theorem 1, *and put* $z = \sum_i x_i T_i, C = A(T)/zA(T)$ *and* $J = IC$. *If* $\overline{J^{n+2}} = J^n\overline{J^2}$ *for every* $n \geq 0$, *then* $\overline{I^{n+2}} = I^n\overline{I^2}$ *for every* $n \geq 0$.

PROPOSITION 11. *Let* r *be either 1 or 2. Then the following assertions are equivalent.*

(1) $\overline{I^{n+r}} = I^n\overline{I^r}$ *for every* $n \geq 0$.
(2) $[H^i_N(R')]_j = 0$ *for* $i + j \geq r + 1$.
(3) $[H^i_N(R')]_j = 0$ *for* $i + j = r + 1$.

Proof of Proposition 10: Assume that $d \geq 3$, and choose a system of generators $x_1, \cdots, x_d$ satisfying the conditions of Theorem 3, and put $z = \sum_i x_i T_i, C = A(T)/zA(T), J = IC$, as in Theorem 3. $S = A(T) \otimes R (= R(A(T), IA(T)))$, $S' = A(T) \otimes R' (= R'(A(T), IA(T)))$, $N' = ItS, M' = (t^{-1}, It)S'$ $F = S/ztS (= R(C, J))$ and $F' = \sum \overline{I^n C}t^n (= R'(C, J))$. Suppose that $[H^i_{N'}(F')]_j = 0$ for $i + j = 3$. We shall prove $[H^i_N(R')]_j = 0$ for $i + j = 3$. Since $A \longrightarrow A(T)$ is faithfully flat, $H^i_N(R') = H^i_{N'}(S')$; thus it is sufficient to prove that $[H^i_{N'}(S')]_j = 0$ for $i + j = 3$. We first prove that $[H^{i-1}_{N'}(S'/ztS')]_{j+1} = 0$ for $3 \leq i \leq d$ and $j \geq 3 - i$. If this is proved, then $([H^i_N(S')]_j =)[H^i_{N'}(S'(-1))]_{j+1} \xrightarrow{zt} [H^i_{N'}(S')]_{j+1}$ is injective; since every element of $H^i_{N'}(S')$ is annihilated by some power of zt, $[H^i_N(S')]_j$ must be 0. Since $dimF'/(S'/ztS') = 0$, we have $H^1_{N'}(S'/ztS') = F'/(S'/ztS')$ and $H^i_{N'}(S'/ztS') = H^i_{N'}(F')$ for $i \geq 2$. Therefore, for $3 \leq i \leq d$, $H^{i-1}_{N'}(S'/ztS')_{j+1} = H^{i-1}_{N'}(F')_{j+1} = 0$, since F' is a Cohen-Macaulay ring. We next prove that $H^2_{N'}(S')_1 = 0$. It is known that $H^2_{N'}(S')_0 = 0$; therefore

$0 = H^2_{N'}(S')_0 \xrightarrow{zt} H^2_{N'}(S')_1 \longrightarrow H^2_{N'}(S'/ztS')_1 = 0$; hence $H^2_{N'}(S')1 = 0$. It is also known that $H^0_{M'}(S') = H^1_{M'}(S') = 0$ and $H^i_{M'}(S') = H^i_{N'}(S')i = 0, 1$. Therefore $H^i_{N'}(S') = 0$ for $i = 0, 1$.

4. $\overline{e}_3(I)$.

In general, it is known that

$$(-1)^d \overline{e}_d(I) = F(0) = \sum_i (-1)^i length_A([H^i_N(R')]_0).$$

If $d = 3$, then $\overline{e}_3(I) = length_A([H^3_N(R')]_0) = length_A(H^2(X, O_X))$, because $[H^2_N(R')]_0 = [H^2_M(R')]_0 = 0$. Therefore by Proposition 5,

(***) $\overline{e}_3(I) \geq 0$.

It is then natural to ask when $\overline{e}_3 = 0$. It follows from [4, Appendix 2] that there exists a canonical graded homomorphism $\alpha : H^d_N(R') \longrightarrow H^d_m(A)[t, t^{-1}]$. We denote by α_j the graded part of degree j of α. Then we have

LEMMA 12. *$\alpha_j = 0$ (i.e., $[H^d_M(R')]_j = [H^d_N(R')]_j$) if and only if $\overline{I^{n+d-1+j}} \subseteq I^n$ for every $n \geq 0$.*

Proof of Theorem 2. (1): By Proposition 5, we may assume that $d = dimA = 3$. Then the assertion follows from (***) and Lemma 12. (2): This follows from the following proposition.

PROPOSITION 13. *Assume that A is a Gorenstein local ring and $\overline{I^2} = I\overline{I}$. If $\overline{I^{n+2}}$ and $m\overline{I^{n+1}}$ are contained in I^n for every $n \geq 0$, then $length_A(\overline{I^2}/I\overline{I}) = 1$ and $\overline{I^{n+2}} = I^n \overline{I^2}$ for every $n \geq 0$.*

PROOF. Since $m\overline{I^{n+1}} \supseteq I^n$, we have $(I^n : m)/I^n \supseteq (\overline{I^{n+1}} + I^n)/I^n = \overline{I^{n+1}}/I^n\overline{I}$, and hence $length_A(\overline{I^{n+1}}/I^n\overline{I}) \leq length_A((I^n : m)/I^n) = \binom{n-1+d-1}{d-1}$, because A is Gorenstein. Therefore

$$\begin{aligned}
&length_A(A/\overline{I^{n+1}}) \\
&\quad = length_A(A/I^{n+1}) - length_A((I^n\overline{I}/I^{n+1}) - length_A(\overline{I^{n+1}}/I^n\overline{I}) \\
&\quad \geq length_A(A/I)\binom{n+d}{d} - length_A(\overline{I}/I)\binom{n+d-1}{d-1} - \binom{n-1+d-1}{d-1} \\
&\quad = length_A(A/I)\binom{n+d}{d} - (length_A(\overline{I}/I) + 1)\binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}.
\end{aligned}$$

(We have already proved in [4, Proposition 10] that $length_A((I^n\overline{I}/I^{n+1}) = length_A(\overline{I}/I)\binom{n+d-1}{d-1}$.) Thus by (*), $length_A(\overline{I^2}/I\overline{I}) = 1$ and $length_A(A/\overline{I^{n+1}}) = length_A(A/I)\binom{n+d}{d} - (length_A(\overline{I}/I)+1)\binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}$, and in particular, $\overline{I^{n+1}} = I^{n-1}\overline{I^2}$.

It is natural to ask whether the assertion (2) in Theorem 2 is true for any parameter ideals.

CONJECTURE: Assume that A is Gorenstein and $d = dimA \geq 3$. Then $\overline{e}_3(I) = 0$ if and only if $\overline{I^{n+2}} = I^n\overline{I^2}$ for every $n \geq 0$.

Assume that $d = 3$ and $\overline{e}_3(I) = 0$: By Proposition 11, if $[H^2_M(R')]_1 (= [H^2_N(R')]_1) = 0$, then $\overline{I^{n+2}} = I^n\overline{I^2}$ for every $n \geq 0$.

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