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A NOTE ON CERTAIN SUBORDINATIONS

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1. INTRODUCTION

Mocanu([1], Lemma 3) showed the following Lemma 1 in 1986:

LEMMA 1. *If $p(z)$ is an analytic function in $|z|<1$, with $p(0)=1$, and if $\mathrm{Re}\{p(z)+zp'(z)\}>0$ in $|z|<1$, then*

(1) $$\left|\arg p(z)\right| < \frac{\pi}{3}$$

in $|z|<1$.

Here we recall the definition of subordination. Let $p(z)$ and $q(z)$ be analytic functions in $|z|<1$ and $p(0)=q(0)$. $p(z)$ is said to be subordinate to $q(z)$ (written $p(z)\prec q(z)$) if $p(z)=q(w(z))$, $|z|<1$ for some analytic function $w(z)$ with $|w(z)|\leqq|z|$ (See Duren [2], p-190).

Since a function $q(z)=\dfrac{1-z}{1+z}$ satisfies $q(0)=1$ and $\left|\arg q(z)\right| < \dfrac{\pi}{2}$, the function $p(z)$ in Lemma 1 is represented by the subordination,

(2) $$p(z) \prec \left(\frac{1-z}{1+z}\right)^{2/3}$$

In 1987 Miller and Mocanu ([3],Theorem 5) proved the following Lemma 2 by considering a differential subordinate system.

LEMMA 2. *Let* $\beta_0 = 1.218\cdots$ *be the solution of*

$$(3) \qquad \beta \pi \frac{3\pi}{2} - \tan^{-1}\beta$$

and let

$$(4) \qquad \alpha = \alpha(\beta) = \beta + \frac{2}{\pi}\tan^{-1}\beta$$

for $0 < \beta \leqq \beta_0$. *If* $p(z)$ *is an analytic function in* $|z| < 1$, *with* $p(0) = 1$, *then*

$$(5) \qquad p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \Longrightarrow p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

When $\alpha = 1$ in the equality (4), we have

$$(6) \qquad 1 = \beta + \frac{2}{\pi}\tan^{-1}\beta \qquad (0 < \beta \leqq \beta_0),$$

$\beta^* = 0.638\cdots\cdots$, and β^* is the solution of (6). This shows that

$$(7) \qquad Re\{p(z) + z\,p'(z)\} > 0,\ |z| < 1 \quad \Longrightarrow p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}$$

and $\beta^* < \dfrac{2}{3}$. Hence Lemma 1 can be improved by such as the following

LEMMA 3. *If* $p(z)$ *is an analytic function in* $|z| < 1$, *with* $p(0) = 1$ *and* if $Re\{p(z) + z\,p'(z)\} > 0$, $|z| < 1$, *then we have*

$$(7) \qquad \left|\arg p(z)\right| < \frac{\pi}{2}\beta^*,$$

$|z| < 1$, *where* $\beta^* = 0.638\cdots\cdots$, and β^* *is the solution of* (6).

REMARK. It seems that an extremal function in Lemma 3 is

$$(8) \qquad p(z) = \frac{2}{z}\log(1+z) - 1 .$$

This function $p(z) = \frac{2}{z}\log(1+z) - 1$ satisfies

(i) $\quad Re\{p(z)\} > 2\log 2 - 1,$

(ii) $\quad p(z) \prec \left(\frac{1-z}{1+z}\right)^{\beta_1},$

where $\beta_1 = 0.503 \cdots\cdots$. But we can not prove these facts at present.

2. THEOREMS AND PROOFS

From Lemma 3 we can show the following

THEOREM 1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $|z| < 1$ *and* $Re\{f'(z)\} > 0$ *in* $|z| < 1$, *then*

(9) $$\left|\arg \frac{f(z)}{z}\right| = \left|\arg \int_0^r f'(\rho e^{i\theta})\, d\rho\right| < \frac{\pi}{2}\beta^*,$$

where $z = re^{i\theta}$, $0 < r < 1$, *and* β^* *is given in Lemma 3.*

Proof. We put $p(z) = \frac{f(z)}{z}$ in Lemma 3, then we have $p(0) = 1$, and

(10) $$Re\{p(z) + z\,p'(z)\} = Re\{f'(z)\} > 0 \quad \text{in } |z| < 1.$$

Therefore, we obtain the following relations

(11) $$\left|\arg p(z)\right| = \left|\arg \frac{f(z)}{z}\right|$$

$$= \left|\arg \frac{1}{z}\int_0^z f'(t)\, dt\right| = \left|\arg \frac{1}{re^{i\theta}}\int_0^r f'(\rho e^{i\theta})\, e^{i\theta}\, d\rho\right|$$

$$= \left|\arg \frac{1}{r}\int_0^r f'(\rho e^{i\theta})\, d\rho\right|$$

$$= \left| \arg \int_0^r f'(\rho e^{i\theta})\, d\rho \right| < \frac{\pi}{2} \beta^* ,$$

where $z = re^{i\theta}$, $0 < r < 1$. $t = \rho e^{i\theta}$ and $0 \leq \rho \leq r$.

First the function $f(z)$ in Theorem 1 are to be said close-to-convex functions in $|z| < 1$ and next Theorem 2 is a result for convex functions in $|z| < 1$.

THEOREM 2. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $|z| < 1$ *and* $Re\{1 + \frac{zf''(z)}{f'(z)}\} > 0$ *in* $|z| < 1$ *(i.e., f(z) are a univalent and convex functions) then we have*

(12) $$\left| \arg \int_0^r \{1 + \frac{\rho e^{i\theta} f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\}\, d\rho \right| < \frac{\pi}{2} \beta^*,$$

where $z = re^{i\theta}$, $0 < r < 1$. *and* β^* *is given in Lemma 3.*

Proof. We put $q(z) = \frac{zf'(z)}{f(z)}$ and

(13) $$p(z) = \frac{1}{z} \int_0^z \{q(t) + \frac{t\, q'(t)}{q(t)}\} dt ,$$

then we have

(14) $$p(z) + z\, p'(z) = q(z) + \frac{z\, q'(z)}{q(z)} = 1 + \frac{zf''(z)}{f'(z)} ,$$

$p(0) = 1$, $q(0) = 1$ and $p(z)$, $q(z)$ are analytic in $|z| < 1$.
Here we again use a method similar to that in Theorem 1. Since

(15) $$Re\{p(z) + z\, p'(z)\} = Re\{1 + \frac{zf''(z)}{f'(z)}\} > 0 \ , \ |z| < 1$$

by the hypotheses, we have the following relations

(16) $$|\arg p(z)| = \left| \arg \frac{1}{z} \int_0^z \{q(t) + \frac{t\, q'(t)}{q(t)}\} dt \right|$$

$$= \left| \arg \frac{1}{z} \int_0^z \{1 + \frac{tf''(t)}{f'(t)}\} dt \right|$$

$$= \left| \arg \frac{1}{r} \int_0^r \{1 + \frac{\rho e^{i\theta} f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\} d\rho \right|$$

$$= \left| \arg \int_0^r \{1 + \frac{\rho e^{i\theta} f''(\rho e^{i\theta})}{f'(\rho e^{i\theta})}\} d\rho \right| < \frac{\pi}{2} \beta^*,$$

where $z = re^{i\theta}$, $0 < r < 1$, $t = \rho e^{i\theta}$, $0 \leq \rho \leq r$, and β^* is given in Lemma 3. This completes the proof.

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