数理解析研究所講究録
第 714 巻 1990 年 91-109

On Behaviors of Hypergeometric Series of One, Two, Three and n Variables Near Boundaries of Their Convergence Regions of Logarithmic Case

Megumi Saigo

Department of Applied Mathematics, Fukuoka University

Fukuoka 814-01, Japan

西郷　恵　（福岡大・理）

1. Introduction

A lot of properties are already known for various hypergeometric series in the literature. Especially, representation formulas exhibiting behaviors of the series near boundaries of their convergence regions are interesting and applicable to some calculations in [12], [13] and [27].

The present report is intended to summarize the results of the behaviors to distinguishing that for which series the behavior is known and to serve further investigation of such a calculation by finding some new ideas.

The generalized hypergeometric series of one variable is defined by

(1.1) $${}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;x) = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_m}{\prod_{j=1}^{q}(\beta_j)_m} \frac{x^m}{m!},$$

where $(\alpha)_m$ denotes the Pochhammer symbol defined by $\Gamma(\alpha+m)/\Gamma(\alpha)$. The series (1.1) is called Gaussian series if $p = 2$, $q = 1$, and Clausenian series if $p = 3$, $q = 2$. It is well known (see e.g. [8], [26]) that the series (1.1) with $q = p-1$ converges in the region $|x| < 1$ for any complex parameters α_j and $\beta_j \neq$ nonpositive integers, as well as at $x = 1$ for $\mathrm{Re}(m) > 0$, where

$$(1.2) \qquad m = \sum_{j=1}^{p-1} \beta_j - \sum_{j=1}^{p} \alpha_j .$$

We are interesting how the series ${}_pF_{p-1}(x)$ behaves near the point $x = 1$ when $m = 0$, for which the series is called zero-balanced. In fact, Srinivasa Ramanujan already discovered the property for the case $p = 3$ in his Notebook [9, Chapt. 11, Entry 24, Cor. 2] without proof, which may be read in the form:

$$(1.3) \qquad {}_3F_2(a,b,c;d,a+b+c-d;1-\rho)$$

$$= \frac{(a+b-d)(d-c)\Gamma(d)\Gamma(a+b+c-d)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c)} \, {}_4F_3(1,1,a+b-d+1,d-c+1;2,a+1,b+1;1)$$

$$- \frac{\Gamma(d)\Gamma(a+b+c-d)}{\Gamma(a)\Gamma(b)\Gamma(c)} [2\gamma + \psi(a) + \psi(b) + \log \rho] + o(1), \qquad (\rho \to +0),$$

for $\mathrm{Re}(c) > 0$, where $\psi(a)$ means the psi function defined by the logarithmic derivative of the gamma function $\Gamma'(a)/\Gamma(a)$, and $\gamma = -\psi(1)$ is the Euler-Macheroni constant. Berndt express (1.3) to be "a very beautiful and significant formula" in his book [2].

The formula (1.3) implies, if we set $d = c$, the formula for the Gauss series:

$$(1.4)\qquad {}_2F_1(a,b;a+b;1-\rho) = -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}[2\gamma + \psi(a) + \psi(b) + \log \rho] + o(1),$$

$$(\rho \to +0),$$

which is a simple result from the continuation formula for the Gauss function (see [14]).

The formula (1.3) was left long mysterious fact, because Ramanujan did not refer to any proof in his Notebook, and no one could derive from the theorem (Entry 24), though it was displayed as a corollary. About the formula (1.3), Berndt mentioned in [2] as "Corollary 2 in Section 24 offers a certain asymptotic formula for zero-balanced ${}_3F_2$ series. Such formulas in the literature have previously been established only for zero-balanced ${}_2F_1$ series. It is interesting that this elegant formula had been overlooked for 60 years after Ramanujan's death." Recently, Evans and Stanton [5] and Bühring [3] proved (1.3) by their own methods. Independently of their proofs, the author established the formula (1.3) in [14], that was published ealier (1983) than the papers [5](1984) and [3](1987). The author did not know until 1988 that the formula (1.3) was the result by Ramanujan and left long as a mystery. The author's interest was in the direction of some calculations of the multiplication of the fractional integral and derivative which were defined by him as generalizations of that of Riemann-Liouville and Erdélyi-Kober (cf. [10], [12], [13], [27]). During the stay at the University of Victoria in Canada (March, 1988 - February, 1989), the author came to know the formula (1.3) occupied the attention and its proofs appeared in [5] and [3]. Concerning the proofs in [5] and [3], Berndt also wrote in [2] that "It would be interesting to have a more direct proof that might shed some light on Ramanujan's approach." The author is certain that his proof is just the direct one.

Moreover, the author knew at the University of Victoria that a generalization of the formula (1.3) to ${}_pF_{p-1}$ with $p \geq 4$ was posed as an open problem by Evans [4, p.553]. But the problem was solved by us, the author and H.M. Srivastava, in [24] by a similar method with that for $p = 3$ in [14]. The result for $p = 4$ is:

(1.5)
$$
\begin{aligned}
&{}_4F_3(a_1,a_2,a_3,a_4;b_1,b_2,b_3;1-\rho) \\
&= \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1+1)\Gamma(a_2+1)\Gamma(a_3)\Gamma(a_4)} (b_1-a_3)(a_1+a_2-b_1) \\
&\qquad \cdot {}_4F_3(1,1,b_1-a_3+1,a_1+a_2-b_1+1;2,a_1+1,a_2+1;1) \\
&+ \frac{\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_4)\Gamma(b_2+b_3-a_4+1)} (b_2-a_4)(b_3-a_4) \\
&\qquad \cdot F^{0:3;3}_{1:1;1}\left[\begin{matrix} \text{———} : a_1,a_2,a_3; b_2-a_4+1,b_3-a_4+1,1; \\ b_2+b_3-a_4+1: \quad b_1; \quad 2; \end{matrix} \; 1,1\right] \\
&- \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)} [2\gamma + \psi(a_1) + \psi(a_2) + \log\rho] + o(1), \\
&\qquad\qquad (\rho \to +0),
\end{aligned}
$$

where $a_1+a_2+a_3+a_4 = b_1+b_2+b_3$, $\mathrm{Re}(a_3) > 0$, $\mathrm{Re}(a_4) > 0$, and $F^{0:3;3}_{1:1;1}$ means the Kampé de Fériet function (see e.g. [1], [26]). The formula (1.5) implies (1.3) if $a_4 = b_3$.

For general, if $p \geq 5$ we have

(1.6) $${}_pF_{p-1}(a_1,\cdots,a_p;b_1,\cdots,b_{p-1};1-\rho)$$

$$= \frac{\Gamma(b_1)\cdots\Gamma(b_{p-1})}{\Gamma(a_1)\cdots\Gamma(a_p)}\ \frac{b_1-a_3}{a_1a_2}\left\{\sum_{j=2}^{p-1} b_j - \sum_{j=3}^{p} a_j\right\}$$

$$\cdot\ {}_4F_3\left(1,1,b_1-a_3+1,\sum_{j=2}^{p-1} b_j - \sum_{j=3}^{p} a_j +1;2,a_1+1,a_2+1;1\right)$$

$$+\sum_{k=3}^{p-1}\frac{\Gamma(b_{k-1})\cdots\Gamma(b_{p-1})}{\Gamma(a_{k+1})\cdots\Gamma(a_p)}\ \frac{(b_{k-1}-a_{k+1})\left(\sum_{j=k}^{p-1} b_j - \sum_{j=k+1}^{p} a_j\right)}{\Gamma\left(\sum_{j=k-1}^{p-1} b_j - \sum_{j=k+1}^{p} a_j +1\right)}$$

$$\cdot\ F^{0:\ \ k;3}_{1:k-2;1}\left[\begin{matrix} \rule{3cm}{0.4pt} :a_1,\cdots,a_k\ ;b_{k-1}-a_{k+1}+1,\sum_{j=k}^{p-1} b_j-\sum_{j=k+1}^{p} a_j+1,1; \\ \sum_{j=k-1}^{p-1} b_j-\sum_{j=k+1}^{p} a_j+1:b_1,\cdots,b_{k-2}; \qquad\qquad 2; \end{matrix}\ 1,1\right]$$

$$-\frac{\Gamma(b_1)\cdots\Gamma(b_{p-1})}{\Gamma(a_1)\cdots\Gamma(a_p)}\ [2\gamma+\psi(a_1)+\psi(a_2)+\log\rho]+o(1),$$

$$(\rho\to+0),$$

where $a_1+\cdots+a_p = b_1+\cdots+b_{p-1}$, $\mathrm{Re}(a_j) > 0$, $(j = 3,\cdots,p)$.

We introduce in the next section various hypergeometric series which are interesting to discuss through our methods, and their convergence regions of three variables are figured. For their precise definitions and the demonstration of their convergence regions, we refer to [1], [6], [7], [8] and [26]. Section 3 is devoted to mention the results obtained up to the present.

2. Hypergeometric Series and Their Convergence Regions

Single variable: As indicated in the previous section, the series ${}_pF_{p-1}(x)$ has the region of convergence $|x| < 1$ for any integer $p \geq 2$.

Two variables: The Appell series of two variables are as follows:

$F_1(a,b_1,b_2;c;x,y)$ in $|x| < 1,\ |y| < 1$

$F_2(a,b_1,b_2;c_1,c_2;x,y)$ in $|x| + |y| < 1$

$F_3(a_1,a_2,b_1,b_2;c;x,y)$ in $|x| < 1,\ |y| < 1$

$F_4(a,b;c_1,c_2;x,y)$ in $\sqrt{|x|} + \sqrt{|y|} < 1$

Three variables: Let $|x| = r$, $|y| = s$, and $|z| = t$, the convergence regions of triple hypergeometric series of Lauricella and Srivastava are shown as follows with figures:

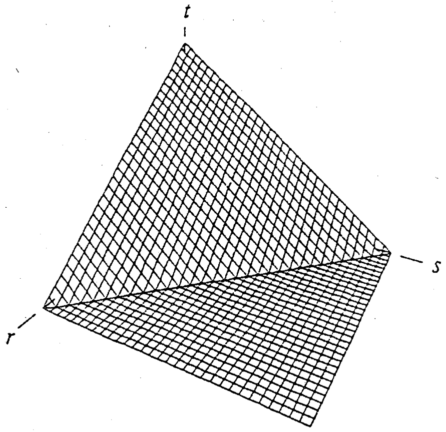
$F_A(a,a,a,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z)$ $r+s+t < 1$

$F_B(a_1,a_2,a_3,b_1,b_2,b_3;c,c,c;x,y,z)$ $r < 1,\ s < 1,\ t < 1$

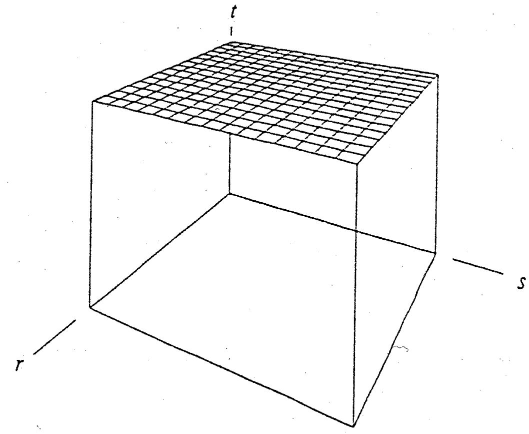
$F_C(a,a,a,b,b,b;c_1,c_2,c_3;x,y,z)$ $\sqrt{r}+\sqrt{s}+\sqrt{t} < 1$

$F_D(a,a,a,b_1,b_2,b_3;c,c,c;x,y,z)$ $r < 1,\ s < 1,\ t < 1$

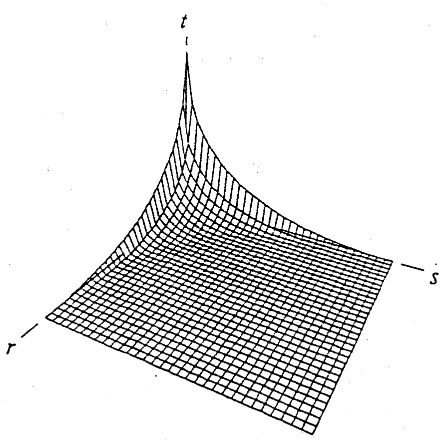
$F_E(a,a,a,b_1,b_2,b_2;c_1,c_2,c_3;x,y,z)$	$r < 1-(\sqrt{s}+\sqrt{t})^2$
$F_F(a,a,a,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z)$	$\sqrt{r}+\sqrt{t} < 1,\ s < \frac{1}{2}(1+t-r+\sqrt{(1+t-r)^2-4t})$
$F_G(a,a,a,b_1,b_2,b_3;c_1,c_2,c_2;x,y,z)$	$r+s < 1,\ r+t < 1$
$F_K(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;x,y,z)$	$r < 1,\ s < 1,\ t < (1-s)(1-r)$
$F_M(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z)$	$r+t < 1,\ s < 1$
$F_N(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z)$	$r+t < 1,\ s < 1$
$F_P(a_1,a_2,a_1,b_1,b_1,b_2;c_1,c_2,c_2;x,y,z)$	$r+s < 1,\ r+t < 1$
$F_R(a_1,a_2,a_1,b_1,b_2,b_1;c_1,c_2,c_2;x,y,z)$	$\sqrt{r}+\sqrt{t} < 1,\ s < 1$
$F_S(a_1,a_2,a_2,b_1,b_2,b_3;c,c,c;x,y,z)$	$r < 1,\ s < 1,\ t < 1$
$F_T(a_1,a_2,a_2,b_1,b_2,b_1;c,c,c;x,y,z)$	$r < 1,\ s < 1,\ t < 1$
$H_A(a_1,a_2,a_3;c_1,c_2;x,y,z)$	$r < (1-s)(1-t),\ s < 1,\ t < 1$
$H_B(a_1,a_2,a_3;c_1,c_2,c_3;x,y,z)$	$r+s+t+2\sqrt{rst} < 1$
$H_C(a_1,a_2,a_3;c;x,y,z)$	$r < 1,\ s < 1,\ t < 1,\ r+s+t-2\sqrt{(1-r)(1-s)(1-t)} < 2$



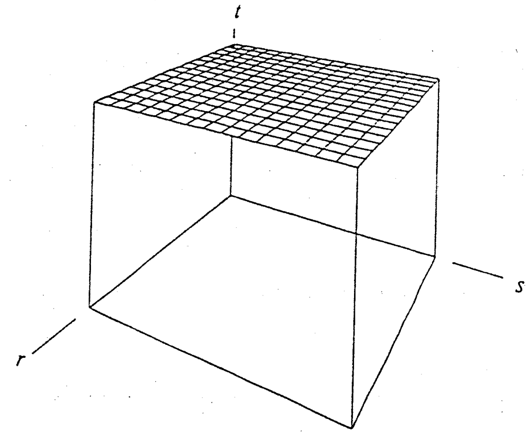
F_A : $r + s + t < 1$



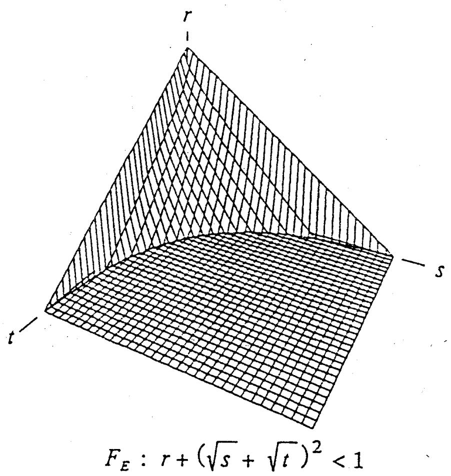
F_B : $r < 1,\ s < 1,\ t < 1$



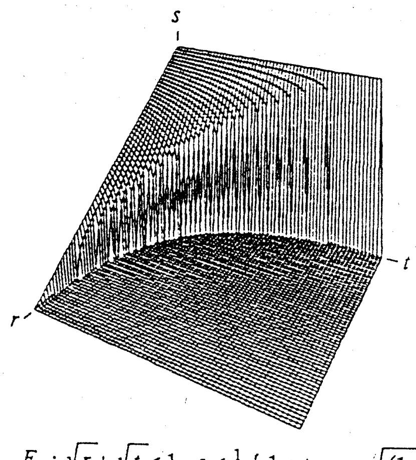
F_C : $\sqrt{r} + \sqrt{s} + \sqrt{t} < 1$



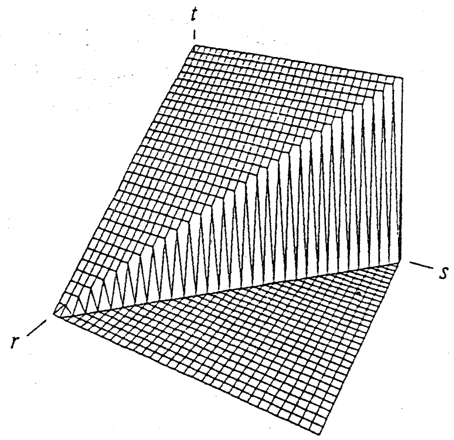
F_D : $r < 1,\ s < 1,\ t < 1$



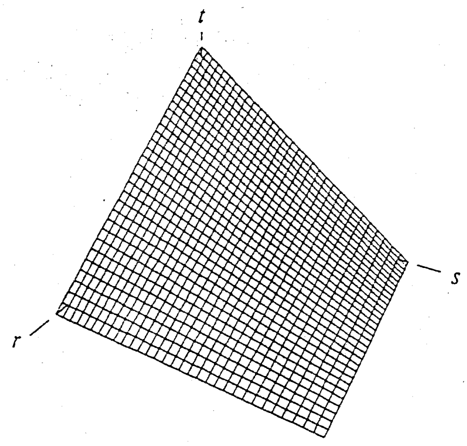
F_E : $r + (\sqrt{s} + \sqrt{t})^2 < 1$



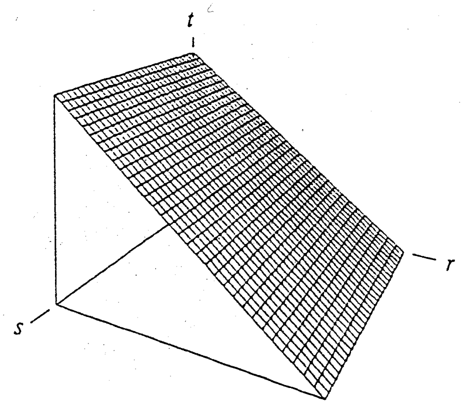
F_F : $\sqrt{r} + \sqrt{t} < 1,\ s < \frac{1}{2}\{1 + t - r + \sqrt{(1+t-r)^2 - 4t}\}$



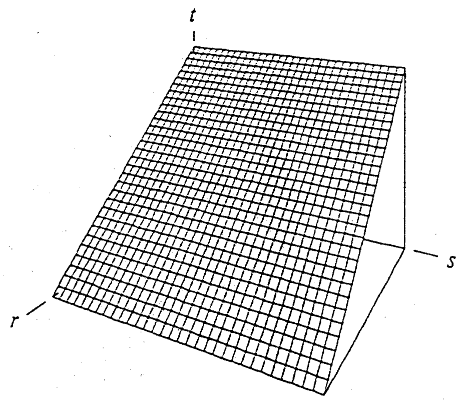
F_G: $r + s < 1$, $r + t < 1$



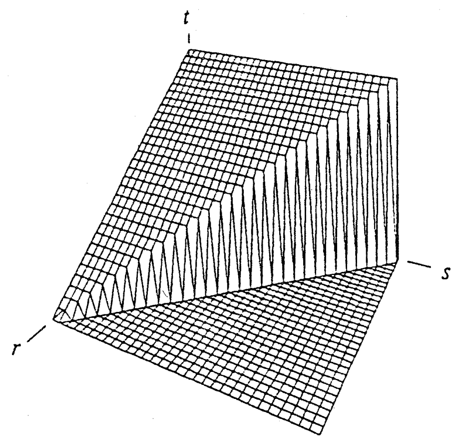
F_K: $r < 1$, $s < 1$, $t < (1-r)(1-s)$



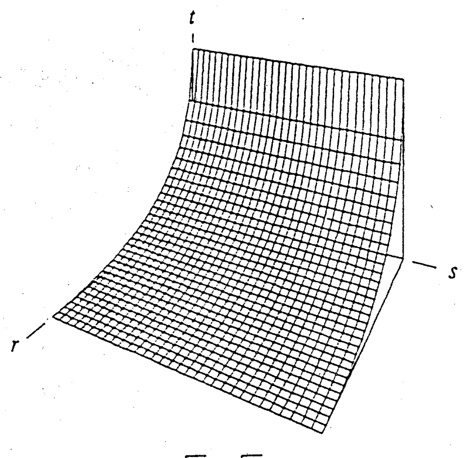
F_M: $r + t < 1$, $s < 1$



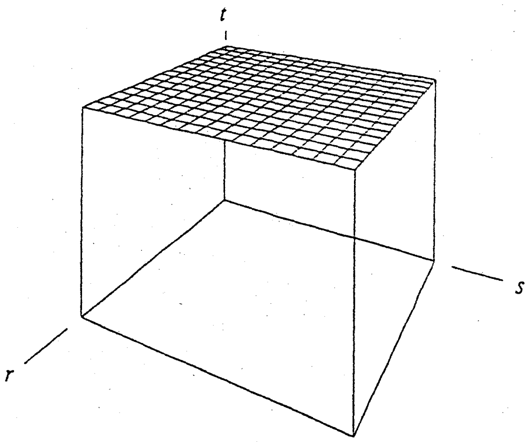
F_N: $r + t < 1$, $s < 1$



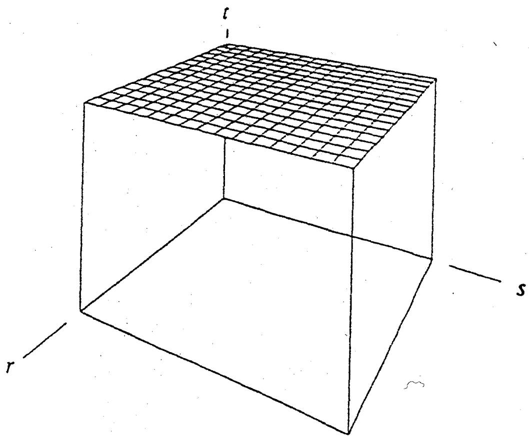
F_P: $r + s < 1$, $r + t < 1$



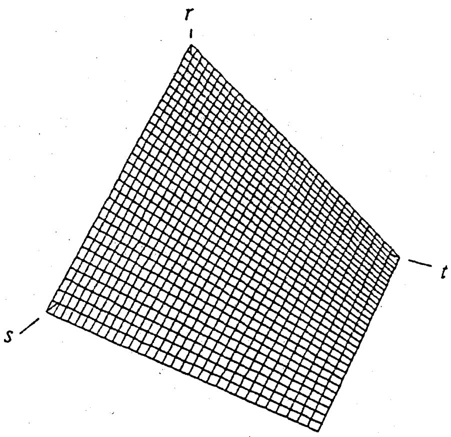
F_R: $\sqrt{r} + \sqrt{t} < 1$, $s < 1$



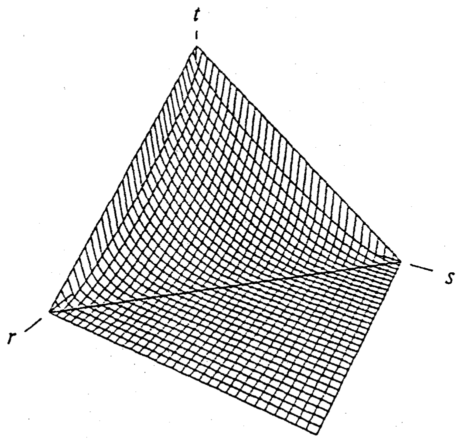
F_S: $r < 1,\ s < 1,\ t < 1$



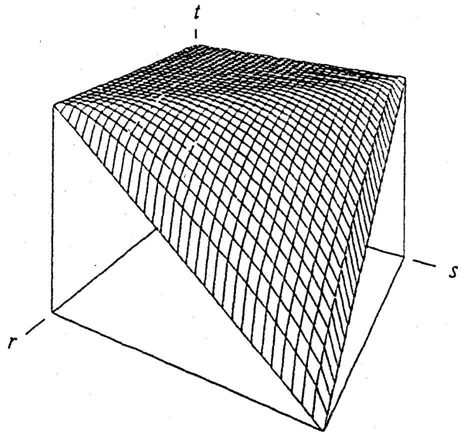
F_T: $r < 1,\ s < 1,\ t < 1$



H_A: $r < (1-s)(1-t),\ s < 1,\ t < 1$



H_B: $r + s + t + 2\sqrt{rst} < 1$



H_C: $r < 1,\ s < 1,\ t < 1,\ r + s + t - 2\sqrt{(1-r)(1-s)(1-t)} < 2$

Multivariables: Let $|x_i| = r_i$, $(i = 1,\cdots,n)$, then the convergence regions of Lauricella hypergeometric series of n variables are as follows:

$F_A^{(n)}(a,b_1,\cdots,b_n;c_1,\cdots,c_n;x_1,\cdots,x_n)$	$r_1+\cdots+r_n < 1$
$F_B^{(n)}(a_1,\cdots,a_n,b_1,\cdots,b_n;c;x_1,\cdots,x_n)$	$r_1 < 1, \cdots, r_n < 1$
$F_C^{(n)}(a,b;c_1,\cdots,c_n;x_1,\cdots,x_n)$	$\sqrt{r_1}+\cdots+\sqrt{r_n} < 1$
$F_D^{(n)}(a,b_1,\cdots,b_n;c;x_1,\cdots,x_n)$	$r_1 < 1, \cdots, r_n < 1$

3. Properties of Hypergeometric Series

We shall state here whole possibility of the boundary points of convergence regions of hypergeometric series introduced in the previous section. The reference numbers indicate that their behaviors were already calculated. The term "complete" means the formula is known with no restriction on parameters without some zero-balancedness, and "2 cases" means that two formulas are evaluated with some other restrictions. The calculations are tried only for the listed Gaussian complete series. The cases which are not indicated the reference numbers are open problems for the reader.

Single variable:

${}_2F_1[1-\rho]$ complete: [14]

${}_3F_2[1-\rho]$ complete: [14]

${}_pF_{p-1}[1-\rho]$ complete: [24]

Two variables:

$F_1[1-\rho,y]$ complete: included in [15]

$F_1[1-\rho x,1-\rho y]$ complete: [11]

$F_2[x,1-x-\rho]$ 2 cases: [14] complete: [20]

$F_3[1-\rho,y]$ complete: [14]

$F_3[1-\rho x,1-\rho y]$ complete: included in [23]

$F_4[x,(1-\sqrt{y}-\rho)^2]$ 3 cases: [21]

Three variables:

$F_A[x,y,1-x-y-\rho]$ complete: included in [18]

$F_B[x,y,1-\rho]$ complete: included in [23]

$F_B[x,1-\rho y,1-\rho z]$ complete: included in [23]

$F_B[1-\rho x,1-\rho y,1-\rho z]$ complete: included in [23]

$F_C[x,y,(1-\sqrt{x}-\sqrt{y}-\rho)^2]$ 1 case: [21]

$F_D[x,y,1-\rho]$ complete: [15]

$F_D[x,1-\rho y,1-\rho z]$ complete: [15]

$F_D[1-\rho x,1-\rho y,1-\rho z]$ complete: [15]

$F_E[1-(\sqrt{y}+\sqrt{z})^2-\rho,y,z]$

$F_F[x,y,(1-\sqrt{x}-\rho)^2]$

$F_F[x,\frac{1}{2}(1-x+z+\sqrt{(1-x+z)^2-4z}-\rho),z]$

$F_F[x,1-\sqrt{x}-\rho y,(1-\sqrt{x}-\rho z)^2]$

$F_G[x,y,1-x-\rho]$ 2 cases: [16] complete: [17]

$F_G[x,1-x-\rho,1-x-\rho]$ complete: [17]

$F_K[x,y,(1-x)(1-y)-\rho]$ 1 case: [17]

$F_M[x,1-\rho,z]$ complete: [16]

$F_M[x,y,1-x-\rho]$ 1 case: [16]

$F_M[x, 1-\rho y, 1-x-\rho z]$

$F_N[x, 1-\rho, z]$ complete: [17]

$F_N[x, y, 1-x-\rho]$ 1 case: [21] complete: [17]

$F_N[x, 1-\rho y, 1-x-\rho z]$

$F_P[x, y, 1-x-\rho]$

$F_P[x, 1-x-\rho y, 1-x-\rho z]$

$F_R[x, y, (1-\sqrt{x}-\rho)^2]$ 1 case: [21]

$F_R[x, 1-\rho, z]$ complete: [17]

$F_R[x, 1-\rho y, (1-\sqrt{x}-\rho z)^2]$

$F_S[x, y, 1-\rho]$ complete: [25]

$F_S[x, 1-\rho y, 1-\rho z]$ complete: [25]

$F_S[1-\rho x, 1-\rho y, 1-\rho z]$ complete: [25]

$F_S[1-\rho, y, z]$ complete: [25]

$F_S[1-\rho x, 1-\rho y, z]$ complete: [25]

$F_T[x,y,1-\rho]$ complete: [25]

$F_T[x,1-\rho y,1-\rho z]$ complete: [25]

$F_T[1-\rho x,1-\rho y,1-\rho z]$

$F_T[1-\rho,y,z]$ complete: [25]

$F_T[1-\rho x,1-\rho y,z]$ complete: [25]

$H_A[(1-y)(1-z)-\rho,y,z]$

$H_B[x,y,z]$, $x+y+z+2\sqrt{xyz} = 1-\rho$

$H_C[1-\rho,y,z]$ complete: [19]

$H_C[1-\rho x,y,1-y-\rho]$

$H_C[x,y,z]$, $x+y+z-2\sqrt{(1-x)(1-y)(1-z)} = 2-\rho$

Multivariables:

$F_A^{(n)}[x_1,\cdots,x_{n-1},1-x_1-\cdots-x_{n-1}-\rho]$ complete: [18]

$F_B^{(n)}[x_1,\cdots,x_k,1-\rho x_{k+1},\cdots,1-\rho x_n]$ complete: [23]

$$F_C^{(n)}[x_1,\cdots,x_{n-1},(\sqrt{|x_1|}+\cdots+\sqrt{|x_{n-1}|})^2-\rho]$$

$F_D^{(n)}[x_1,\cdots,x_k,1-\rho x_{k+1},\cdots,1-\rho x_n]$ complete: [22]

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