

The Asymptotic Behaviors of  
Triple Hypergeometric Series,  $F_G$ ,  $F_K$ ,  $F_N$  and  $F_R$   
Near Boundaries of Their Convergence Regions

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1. Introduction

In the series of papers [3] - [6], [8] - [15], the author and his co-authors have established various asymptotic behaviors of hypergeometric series near boundary points of convergence regions. (see also [7]) The present paper is devoted to produce more six properties {(FG1), (FG2), (FK), (FN1), (FN2) and (FR), below} for the Lauricella series of three variables,  $F_G$ ,  $F_K$ ,  $F_N$  and  $F_R$ , which are defined as follows [16]:

$$(1.1) \quad F_G(\alpha, \alpha, \alpha, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z)$$

$$= \sum_{m, n, p=0}^{\infty} \frac{(\alpha)_{m+n+p} (\beta_1)_m (\beta_2)_n (\beta_3)_p}{(\gamma_1)_m (\gamma_2)_{n+p} m! n! p!} x^m y^n z^p,$$

$$\text{in } |x| + |y| < 1, |x| + |z| < 1,$$

$$(1.2) \quad F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= \sum_{m, n, p=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_{n+p} (\beta_1)_{m+p} (\beta_2)_n}{(\gamma_1)_m (\gamma_2)_n (\gamma_3)_p m! n! p!} x^m y^n z^p,$$

$$\text{in } |x| < 1, |y| < 1, |z| < (1-|x|)(1-|y|),$$

$$(1.3) \quad F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z)$$

$$= \sum_{m, n, p=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_n (\alpha_3)_p (\beta_1)_{m+p} (\beta_2)_n}{(\gamma_1)_m (\gamma_2)_{n+p} m! n! p!} x^m y^n z^p,$$

$$\text{in } |x| + |z| < 1, |y| < 1,$$

$$(1.4) \quad F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z)$$

$$= \sum_{m, n, p=0}^{\infty} \frac{(\alpha_1)_{m+p} (\alpha_2)_n (\beta_1)_{m+p} (\beta_2)_n}{(\gamma_1)_m (\gamma_2)_{n+p} m! n! p!} x^m y^n z^p,$$

$$\text{in } \sqrt{|x|} + \sqrt{|z|} < 1, |y| < 1,$$

where  $(\alpha)_m$  means the usual Pochhammer symbol defined by  $\Gamma(\alpha+m)/\Gamma(\alpha)$ .

In the following establishment we shall make use of the following two formulas for the Gauss series  ${}_2F_1$  [4] and the second Appell series  $F_2$  [10]:

$$(1.5) \quad {}_2F_1(a, b; a+b; 1-p) = -\Gamma \begin{bmatrix} a+b \\ a, b \end{bmatrix} [2\gamma + \psi(a) + \psi(b) + \log p] + o(1),$$

$$(p \rightarrow +0),$$

$$(1.6) \quad F_2(a, b_1, b_2; c_1, c_2; x, \rho, 1-x-\delta)$$

$$= \Gamma \left[ \begin{matrix} c_1, c_1 - a - b_1 \\ c_1 - a, c_1 - b_1 \end{matrix} \right] x^{-a} {}_3F_2 \left[ \begin{matrix} a, a - c_1 + 1, a + b_1 - c_1; \frac{x-1}{x} \\ a + b_1 - c_1 + 1, c_2; x \end{matrix} \right]$$

$$+ \Gamma \left[ \begin{matrix} c_1, c_2 \\ a, b_1 - 1, b_2 + 1 \end{matrix} \right] \frac{c_1 - b_1}{c_1 - a - b_1 + 1} x^{b_1 - c_1 + 1} (1-x)^{c_1 - a - b_1 + 1}$$

$$- {}_4F_3 \left[ \begin{matrix} c_1 - b_1 + 1, 2 - b_1, 1, 1; \frac{x-1}{x} \\ c_1 - a - b_1 + 2, b_2 + 1, 2; x \end{matrix} \right]$$

$$- \Gamma \left[ \begin{matrix} c_1, c_2 \\ a, b_1, b_2 \end{matrix} \right] x^{b_1 - c_1} (1-x)^{c_1 - a - b_1}$$

$$\cdot [2\gamma + \psi(a + b_1 - c_1) + \psi(b_2) - \log(1-x) + \log(\rho + \delta)]$$

$$+ o(1), \quad (\rho \rightarrow +0, \delta \rightarrow +0)$$

with  $a + b_1 + b_2 = c_1 + c_2$ ,  $b_2 \notin \text{integer}$ ,  $c_1 - a - b_1 \notin \text{integer}$  and  $1/2 < x < 1$ , where the generalized hypergeometric series  ${}_pF_q$  of single variable and the second Appell series  $F_2$  of two variables are defined by

$$(1.7) \quad {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = {}_pF_q \left[ \begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^p (\alpha_j)_m}{\prod_{j=1}^q (\beta_j)_m} \frac{z^m}{m!},$$

$$(1.8) \quad F_2(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y) = \sum_{m, n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta_1)_m (\beta_2)_n}{(\gamma_1)_m (\gamma_2)_n m! n!} x^m y^n, \quad |x| + |y| < 1,$$

$\psi$  is the psi function and  $\gamma$  is the Euler-Mascheroni constant. When  $q = p-1$ , (1.7) converges in  $|z| < 1$  (cf. [16]). Here and in what follows, we write, for brevity,

$$\Gamma \left[ \begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \right] = \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_p)}{\Gamma(\beta_1) \cdots \Gamma(\beta_q)}.$$

The formula (1.6) holds valid also for  $0 < x < 1/2$  by exchanging  $b_1$  with  $b_2$  and  $c_1$  with  $c_2$ .

## 2. The Behaviors of the Series $F_G$

Let us first consider the series  $F_G$  defined in (1.1) near the points on the plane  $x + z = 1$  ( $x, z > 0$ ) and on the edge  $x + y = 1$ ,  $x + z = 1$  ( $x, y, z > 0$ ) of the boundary of its convergence region. The results are:

$$(FG1) \quad F_G(a, a, a, b_1, b_2, b_3; c_1, c_2, c_2; x, y, 1-x-p)$$

$$= \Gamma \left[ \begin{matrix} c_2, c_2 - a - b_3 \\ c_2 - a, c_2 - b_3 \end{matrix} \right] (1-x)^{-a} F_{0:2;1}^{1:2;1} \left[ \begin{matrix} a: a - c_2 + 1, a + b_3 - c_2; & b_2; & \frac{x}{x-1}, \frac{y}{1-x} \\ -: & a + b_3 - c_2 + 1, c_1; & c_2 - b_3; & \frac{x}{x-1}, \frac{y}{1-x} \end{matrix} \right]$$

$$+ \Gamma \left[ \begin{matrix} c_1, c_2 \\ a, b_3 - 1, b_1 + 1 \end{matrix} \right] \frac{c_2 - b_3}{c_2 - a - b_3 + 1} x^{c_2 - a - b_3 + 1} (1-x)^{b_3 - c_2 + 1}$$

$$\cdot F_{0:3;1}^{1:3;1} \left[ \begin{matrix} c_2 - b_3 + 1: & & 2 - b_3, 1, 1; & b_2; & \frac{x}{x-1}, \frac{y}{1-x} \\ \text{---} & : & c_2 - a - b_3 + 2, b_1 + 1, 2; & c_2 - b_3; & \frac{x}{x-1}, \frac{y}{1-x} \end{matrix} \right]$$

$$- \Gamma \left[ \begin{matrix} c_1, c_2 \\ a, b_3, b_1 \end{matrix} \right] x^{c_2-a-b_3} (1-x)^{b_2+b_3-c_2} (1-x-y)^{-b_2}$$

$$\cdot [2\gamma + \psi(a+b_3-c_2) + \psi(b_1) - \log x + \log \rho]$$

$$+ o(1), \quad (\rho \rightarrow +0)$$

with  $a+b_1+b_3 = c_1+c_2$ ,  $b_1 \notin \mathbb{Z}$ ,  $a+b_3-c_2 \notin \mathbb{Z}$  and  $0 < x < 1/2$ ,  
 $x < y < 1$ .

**Note.** The formula (FG1) holds valid also for  $1/2 < x < 1$  by exchanging  $b_1$  with  $b_3$  and  $c_1$  with  $c_2$ .

$$(FG2) \quad F_G(a, a, a, b_1, b_2, b_3; c_1, c_2, c_2; x, 1-x-\rho, 1-x-\rho)$$

$$= \Gamma \left[ \begin{matrix} c_1, c_1-a-b_1 \\ c_1-a, c_1-b_1 \end{matrix} \right] x^{-a} {}_3F_2 \left[ \begin{matrix} a, a-c_1+1, a+b_1-c_1; \frac{x-1}{x} \\ a+b_1-c_1+1, c_2; \frac{x-1}{x} \end{matrix} \right]$$

$$+ \Gamma \left[ \begin{matrix} c_1, c_2 \\ a, b_1-1, b_2+b_3+1 \end{matrix} \right] \frac{c_1-b_1}{c_1-a-b_1+1} x^{b_1-c_1+1} (1-x)^{c_1-a-b_1+1}$$

$$\cdot {}_4F_3 \left[ \begin{matrix} c_1-b_1+1, 2-b_1, 1, 1; \frac{x-1}{x} \\ c_1-a-b_1+2, b_2+b_3+1, 2; \frac{x-1}{x} \end{matrix} \right]$$

$$- \Gamma \left[ \begin{matrix} c_1, c_2 \\ a, b_1, b_2+b_3 \end{matrix} \right] x^{b_1-c_1} (1-x)^{c_1-a-b_1}$$

$$[2\gamma + \psi(a+b_1-c_1) + \psi(b_2+b_3) - \log(1-x) + \log \rho] \\ + o(1), \quad (\rho \rightarrow +0)$$

with  $a+b_1+b_2+b_3 = c_1+c_2$ ,  $b_2+b_3 \notin \text{integer}$ ,  $a+b_1-c_1 \notin \text{integer}$  and  $1/2 < x < 1$ .

**Note.** The formula (FG2) holds valid also for  $0 < x < 1/2$  by exchanging  $b_1$  with  $b_2+b_3$  and  $c_1$  with  $c_2$ .

**Proof of (FG1).** Expanding  $F_G$  in the series of  $y$ , we have its representation in terms with  $F_2$ :

$$(2.1) \quad F_G(\alpha, \alpha, \alpha, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) \\ = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta_2)_m}{(\gamma_2)_m m!} F_2(\alpha+m, \beta_1, \beta_3; \gamma_1, \gamma_2+m; x, z) y^m$$

Then the relations (2.1) and (1.6) lead us to for  $a+b_1+b_3 = c_1+c_2$

$$(2.2) \quad F_G(a, a, a, b_1, b_2, b_3; c_1, c_2, c_2; x, y, 1-x-\rho) \\ = \sum_{m=0}^{\infty} \frac{(a)_m (b_2)_m}{(c_2)_m m!} F_2(a+m, b_1, b_3; c_1, c_2+m; x, 1-x-\rho) y^m \\ = \sum_{m=0}^{\infty} \frac{(a)_m (b_2)_m}{(c_2)_m m!} y^m \left\{ \Gamma \left[ \begin{matrix} c_2+m, c_2-a-b_3 \\ c_2-a, c_2-b_3+m \end{matrix} \right] (1-x)^{-a-m} \right.$$

$$\begin{aligned}
& \cdot {}_3F_2 \left[ \begin{matrix} a+m, a-c_2+1, a+b_3-c_2; \\ a+b_3-c_2+1, c_1; \end{matrix} \frac{x}{x-1} \right] \\
& + \Gamma \left[ \begin{matrix} c_2+m, c_1 \\ a+m, b_3-1, b_1+1 \end{matrix} \right] \frac{c_2-b_3+m}{c_2-a-b_3+1} x^{c_2-a-b_3+1} (1-x)^{b_3-c_2-m+1} \\
& \cdot {}_4F_3 \left[ \begin{matrix} c_2-b_3+1+m, 2-b_3, 1, 1; \\ c_2-a-b_3+2, b_1+1, 2; \end{matrix} \frac{x}{x-1} \right] \\
& - \Gamma \left[ \begin{matrix} c_2+m, c_1 \\ a+m, b_3, b_1 \end{matrix} \right] x^{c_2-a-b_3} (1-x)^{b_3-c_2-m} \\
& \cdot [2\gamma + \psi(a+b_3-c_2) + \psi(b_1) - \log x + \log \rho] \Big\} \\
& + o(1)
\end{aligned}$$

which deduces the formula (FG1) in terms of the Kampé de Fériet series (cf. [1], [16]):

$$\begin{aligned}
(2.3) \quad & {}_F \begin{matrix} p:r;u \\ q:s;v \end{matrix} \left[ \begin{matrix} \alpha_1, \dots, \alpha_p; \gamma_1, \dots, \gamma_r; \eta_1, \dots, \eta_u; \\ \beta_1, \dots, \beta_q; \delta_1, \dots, \delta_s; \zeta_1, \dots, \zeta_v; \end{matrix} x, y \right] \\
& = \sum_{m, n=0}^{\infty} \frac{\prod_{j=1}^p (\alpha_j)_{m+n} \prod_{j=1}^r (\gamma_j)_m \prod_{j=1}^u (\eta_j)_n}{\prod_{j=1}^q (\beta_j)_{m+n} \prod_{j=1}^s (\delta_j)_m \prod_{j=1}^v (\zeta_j)_n} \frac{x^m y^n}{m! n!}
\end{aligned}$$

Here, the Kampé de Fériet series  $F_{0:2;1}^{1:2;1}[x, y]$  and  $F_{0:3;1}^{1:3;1}[x, y]$  in (FG1) converge in  $|x| + |y| < 1$ .

**Proof of (FG2).** By virtue of the series expansion of  $F_G$  in terms of the Appell series  $F_1$ :

$$(2.4) \quad F_G(\alpha, \alpha, \alpha, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) \\ = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta_1)_m}{(\gamma_1)_m m!} F_1(\alpha+m, \beta_2, \beta_3; \gamma_2; y, z) x^m$$

and by the relation [1]

$$(2.5) \quad F_1(\alpha, \beta_1, \beta_2; \gamma; x, x) = {}_2F_1(\alpha, \beta_1 + \beta_2; \gamma; x),$$

we have the formula

$$(2.6) \quad F_G(\alpha, \alpha, \alpha, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, y) = F_2(\alpha, \beta_1, \beta_2 + \beta_3; \gamma_1, \gamma_2; x, y),$$

where the Appell series  $F_1$  is defined by [1]

$$(2.7) \quad F_1(\alpha, \beta_1, \beta_2; \gamma; x, y) = \sum_{m, n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta_1)_m (\beta_2)_n}{(\gamma)_m m! n!} x^m y^n.$$

Thus the relation (FG2) follows immediately from (1.6), (2.7) and [1]

$$(2.8) \quad F_2(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta_1)_m}{(\gamma_1)_m m!} {}_2F_1(\alpha+m, \beta_2; \gamma_2; y) x^m.$$



### 3. The Behavior of the Series $F_K$

Now let us treat the series  $F_K$  near the surface  $z = (1-x)(1-y)$  ( $x, y, z > 0$ ), then we have:

$$(FK) \quad F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, b_2, c_3; x, y, (1-x)(1-y)^\rho)$$

$$= \Gamma \left[ \begin{matrix} c_1, c_1 - a_1 - b_1 \\ c_1 - a_1, c_1 - b_1 \end{matrix} \right] x^{-b_1} {}_3F_2 \left[ \begin{matrix} b_1, b_1 - c_1 + 1, a_1 + b_1 - c_1; \frac{x-1}{x} \\ a_1 + b_1 - c_1 + 1, c_3; x \end{matrix} \right]$$

$$+ \Gamma \left[ \begin{matrix} c_1, c_3 \\ b_1, a_1 - 1, a_2 + 1 \end{matrix} \right] \frac{c_1 - a_1}{c_1 - a_1 - b_1 + 1} x^{a_1 - c_1 + 1} (1-x)^{c_1 - a_1 + b_1 + 1}$$

$${}^4F_3 \left[ \begin{matrix} c_1 - a_1 + 1, 2 - a_1, 1, 1; \frac{x-1}{x} \\ c_1 - a_1 - b_1 + 2, a_2 + 1, 2; x \end{matrix} \right]$$

$$- \Gamma \left[ \begin{matrix} c_1, c_3 \\ a_1, a_2, b_1 \end{matrix} \right] x^{a_1 - c_1} (1-x)^{c_1 - a_1 - b_1}$$

$$\cdot [2\gamma + \psi(a_1 + b_1 - c_1) + \psi(a_2) - \log(1-x) - \log(1-y) + \log \rho]$$

$$+ o(1), \quad (\rho \rightarrow +0)$$

with  $a_1 + a_2 + b_1 = c_1 + c_3$ ,  $a_2 \neq$  integer,  $a_1 + b_1 - c_1 \neq$  integer and  $1/2 < x < 1$ ,  $0 < y < 1$ .

**Note.** The formula (FG2) holds valid also for  $0 < x < 1/2$  by exchanging  $a_1$  with  $a_2$  and  $c_1$  with  $c_3$ .

**Proof.** The series  $F_K$  may be expanded by the Gauss series as

$$(3.1) \quad F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z) \\ = \sum_{m, n=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_n (\beta_1)_{m+n}}{(\gamma_1)_m (\gamma_3)_n m! n!} {}_2F_1(\alpha_2+m, \beta_2; \gamma_2; y) x^m z^n.$$

Then by noting the relation [2]

$$(3.2) \quad {}_2F_1(\alpha, \beta; \beta; z) = (1-z)^{-\alpha},$$

we have the formula

$$(3.3) \quad F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \beta_2, \gamma_3; x, y, z) \\ = (1-y)^{-\alpha_2} F_2(\beta_1; \alpha_1, \alpha_2; \gamma_1, \gamma_3; x, \frac{z}{1-y}),$$

which together with the relation (1.6) leads us to (FK).

#### 4. The Behaviors of the Series $F_N$

Following two formulas for  $F_N$  defined in (1.3) will be derived:

$$(FN1) \quad F_N(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; x, y, 1-x-\rho) \\ = \Gamma \left[ \begin{matrix} c_1, c_1 - a_1 - b_1 \\ c_1 - a_1, c_1 - b_1 \end{matrix} \right] x^{-b_1} {}_3F_2 \left[ \begin{matrix} b_1, b_1 - c_1 + 1, a_1 + b_1 - c_1; \frac{x-1}{x} \\ a_1 + b_1 - c_1 + 1, c_2; x \end{matrix} \right]$$

$$\begin{aligned}
& + \Gamma \left[ \begin{matrix} c_1, c_2 \\ b_1, a_1-1, a_3+1 \end{matrix} \right] \frac{c_1-a_1}{c_1-a_1-b_1+1} x^{a_1-c_1+1} (1-x)^{c_1-a_1-b_1+1} \\
& \cdot {}_4F_3 \left[ \begin{matrix} c_1-a_1+1, 2-a_1, 1, 1; \frac{x-1}{x} \\ c_1-a_1-b_1+2, a_3+1, 2; x \end{matrix} \right] \\
& + \frac{a_2 b_2}{c_2} y {}_F(3) \left[ \begin{matrix} -:::-; -; b_1: a_1, a_2+1, b_2+1, 1; a_3; \\ -:::-; c_2+1; -: c_1; \quad \quad \quad 2; -; \end{matrix} \right] x, y, 1-x \\
& - \Gamma \left[ \begin{matrix} c_1, c_2 \\ a_1, a_3, b_1 \end{matrix} \right] x^{a_1-c_1} (1-x)^{c_1-a_1-b_1} \\
& \cdot [2\gamma + \psi(a_1+b_1-c_1) + \psi(a_3) - \log(1-x) + \log p] \\
& + o(1), \quad (p \rightarrow +0)
\end{aligned}$$

with  $a_1+a_3+b_1 = c_1+c_2$ ,  $a_3 \notin \text{integer}$ ,  $a_1+b_1-c_1 \notin \text{integer}$  and  $1/2 < x < 1$ ,  $0 < y < 1$ .

**Note.** The formula (FN1) holds valid also for  $0 < x < 1/2$  by exchanging  $a_1$  with  $a_3$  and  $c_1$  with  $c_2$ .

$$(FN2) \quad F_N(a_1, a_2, a_3, b_1, b_2, b_1; c_1, a_2+b_2, a_2+b_2; x, 1-p, z)$$

$$= \Gamma \left[ \begin{matrix} a_2+b_2 \\ a_2, b_2 \end{matrix} \right] \frac{a_3 b_1}{a_2 b_2} z {}_F \begin{matrix} 1:1; 3 \\ 0:1; 3 \end{matrix} \left[ \begin{matrix} b_1+1: a_1; \quad a_3+1, 1, 1; \\ - : c_1; a_2+1, b_2+1, 2; \end{matrix} \right] x, z$$

$$- \Gamma \left[ \begin{matrix} a_2 + b_2 \\ a_2, b_2 \end{matrix} \right] {}_2F_1(a_1, b_1; c_1; x) [2\gamma + \psi(a_2) + \psi(b_2) + \log \rho]$$

$$+ o(1), \quad (\rho \rightarrow +0).$$

**Proof of (FN1).** By the series representation of  $F_N$ :

$$(4.1) \quad F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_2)_m (\beta_2)_m}{(\gamma_2)_m m!} F_2(\beta_1, \alpha_1, \alpha_3; \gamma_1, \gamma_2 + m; x, z) y^m$$

$$= F_2(\beta_1; \alpha_1, \alpha_3; \gamma_1, \gamma_2; x, z)$$

$$+ \frac{\alpha_2 \beta_2}{\gamma_2} y \sum_{m=0}^{\infty} \frac{(\alpha_2 + 1)_m (\beta_2 + 1)_m}{(\gamma_2 + 1)_m (2)_m} F_2(\beta_1, \alpha_1, \alpha_3; \gamma_1, \gamma_2 + 1 + m; x, z) y^m$$

and the relation (1.6), the formula (FN1) follows, where the Srivastava series

$F^{(3)}$  is defined by [16]:

$$(4.2) \quad F^{(3)} \left[ \begin{matrix} (\alpha) :: (\beta^1); (\beta^2); (\beta^3) : (\gamma^1); (\gamma^2); (\gamma^3); \\ (\delta) :: (\eta^1); (\eta^2); (\eta^3) : (\zeta^1); (\zeta^2); (\zeta^3); \end{matrix} ; x, y, z \right]$$

$$= \sum_{m, n, p=0}^{\infty} \Lambda(m, n, p) \frac{x^m y^n z^p}{m! n! p!},$$

where

$$\Lambda(m, n, p) = \frac{\prod_{j=1}^A (\alpha_j)_{m+n+p} \prod_{j=1}^{B_1} (\beta_j^1)_{m+n} \prod_{j=1}^{B_2} (\beta_j^2)_{n+p} \prod_{j=1}^{B_3} (\beta_j^3)_{p+m}}{\prod_{j=1}^D (\delta_j)_{m+n+p} \prod_{j=1}^{E_1} (\eta_j^1)_{m+n} \prod_{j=1}^{E_2} (\eta_j^2)_{n+p} \prod_{j=1}^{E_3} (\eta_j^3)_{p+m}} \cdot \frac{\prod_{j=1}^{C_1} (\gamma_j^1)_m \prod_{j=1}^{C_2} (\gamma_j^2)_n \prod_{j=1}^{C_3} (\gamma_j^3)_p}{\prod_{j=1}^{F_1} (\zeta_j^1)_m \prod_{j=1}^{F_2} (\zeta_j^2)_n \prod_{j=1}^{F_3} (\zeta_j^3)_p},$$

and  $(\alpha)$  abbreviates the set of  $A$  parameters  $\alpha_1, \dots, \alpha_A$  and similar interpretations for  $(\beta_1), (\beta_2)$  etc. Here, the Srivastava series  $F^{(3)}$  in (FN1) converges in  $|x| + |z| < 1, |y| < 1$ .

**Proof of (FN2).** The series expansion

$$(4.3) \quad F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) \\ = \sum_{m, n=0}^{\infty} \frac{(\alpha_1)_m (\alpha_3)_n (\beta_1)_{m+n}}{(\gamma_1)_m (\gamma_2)_n m! n!} {}_2F_1(\alpha_2, \beta_2; \gamma_2^{+n}; y) x^m z^n$$

and (1.5) imply the formula (FN2), where the Kampé de Fériet series  $F_{0:1;3}^{1:1;3}[x, y]$  converges in  $|x| + |y| < 1$ .

### 5. The Behavior of the Series $F_R$

For the series  $F_R$  we have:

$$\begin{aligned}
 (\text{FR}) \quad & F_R(a_1, a_2, a_1, b_1, b_2, b_1; c_1, a_2+b_2, a_2+b_2; x, 1-\rho, z) \\
 &= \Gamma \left[ \begin{matrix} a_2+b_2 \\ a_2, b_2 \end{matrix} \right] \frac{a_1 b_1}{a_2 b_2} z {}_2F_{0:1;3} \left[ \begin{matrix} a_1+1, b_1+1: -; & 1, 1; \\ & c_1; a_2+1, b_2+1, 2; \end{matrix} \right] x, z \\
 &- \Gamma \left[ \begin{matrix} a_2+b_2 \\ a_2, b_2 \end{matrix} \right] {}_2F_1(a_1, b_1; c_1; x) [2\gamma + \psi(a_2) + \psi(b_2) + \log \rho] \\
 &+ o(1), \quad (\rho \rightarrow +0).
 \end{aligned}$$

**Proof.** By noting the series representation of  $F_R$ :

$$\begin{aligned}
 (5.1) \quad & F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) \\
 &= \sum_{m, n=0}^{\infty} \frac{(\alpha_1)_{m+n} (\beta_1)_{m+n}}{(\gamma_1)_m (\gamma_2)_n m! n!} {}_2F_1(\alpha_2, \beta_2; \gamma_2+n; y) x^m z^n
 \end{aligned}$$

we have

$$\begin{aligned}
 & F_R(a_1, a_2, a_1, b_1, b_2, b_1; c_1, a_2+b_2, a_2+b_2; x, 1-\rho, z) \\
 &= {}_2F_1(a_2, b_2; a_2+b_2; 1-\rho) \sum_{m=0}^{\infty} \frac{(a_1)_m (b_1)_m}{(c_1)_m m!} x^m
 \end{aligned}$$

$$+ \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n+1} (b_1)_{m+n+1}}{(c_1)_m (a_2+b_2)_{n+1} m! (n+1)!} {}_2F_1(a_2, b_2; a_2+b_2+n+1; 1-\rho) x^m z^{n+1},$$

which implies (FR) by virtue of (1.5) and the Gauss formula [2]

$$(5.2) \quad {}_2F_1(\alpha, \beta; \gamma; 1) = \Gamma \left[ \begin{array}{c} \gamma, \gamma - \alpha - \beta \\ \gamma - \alpha, \gamma - \beta \end{array} \right] \quad \text{for } \operatorname{Re}(\gamma - \alpha - \beta) > 0.$$

Here, the Kampé de Fériet series  $F_{0:1;3}^{2:0;2}[x, y]$  in (FR) converges in  $\sqrt{|x|} + \sqrt{|y|} < 1$ .

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