

ON AN INVARIANCE PROPERTY OF CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT

The object of the present paper is to link MacGregor's theorem to Pommerenke's one, together.

1. Introduction

MacGregor [2] extended the theorem which was obtained by Sakaguchi [4] and Libera [1] as the following:

Suppose that the function $f(z)$ and $h(z)$ are analytic in $E = \{z \mid |z| < 1\}$, $f(0)=h(0)=0$ and $h(z)$ maps E onto a region which is starlike with respect to the origin. If

$$\operatorname{Re} \frac{f'(z)}{h'(z)} > \beta \quad \text{in } E,$$

then

$$\operatorname{Re} \frac{f(z)}{h(z)} > \beta \quad \text{in } E.$$

On the other hand, Pommerenke [3] proved that if $f(z)$ is analytic in E , $h(z)$ is convex in E and

$$\left| \arg \frac{f'(z)}{h'(z)} \right| \leq \frac{\alpha\pi}{2} \quad \text{in } E$$

where $0 \leq \alpha \leq 1$, then we have

$$\left| \arg \frac{f(z'') - f(z')}{h(z'') - h(z')} \right| \leq \frac{\alpha\pi}{2} \quad \text{in } E$$

where $|z'| < 1$ and $|z''| < 1$.

2. Main theorem

MAIN THEOREM. Let $f(z)$ be analytic in E , $h(z)$ be convex in E and suppose that

$$(1) \quad \left| \arg \left(\frac{f'(z)}{h'(z)} - \beta \right) \right| \leq \frac{\alpha\pi}{2} \quad \text{in } E$$

where $0 \leq \alpha \leq 1$.

Then we have

$$| \arg \left(\frac{f(z'') - f(z')}{h(z'') - h(z')} - \beta \right) | \leq \frac{\alpha\pi}{2} \quad \text{in } E$$

where $|z'| < 1$ and $|z''| < 1$.

PROOF. We will use the same method as in the proof of [3, Lemma 1].

Let $z=h^{-1}(w)$ be the inverse function $w=h(z)$. Then $h^{-1}(w)$ is analytic in the convex domain $h(E)=\{w \mid w=h(z), |z| < 1\}$.

$$\begin{aligned} (2) \quad \frac{f(z'') - f(z')}{h(z'') - h(z')} - \beta &= \frac{f(h^{-1}(w'')) - f(h^{-1}(w'))}{w'' - w'} - \beta \\ &= \int_0^1 [g'(w' + (w'' - w')t) - \beta] dt \end{aligned}$$

where $g(w)=f(h^{-1}(w))$, $w'=h(z')$ and $w''=h(z'')$. On the other hand, we have

$$\begin{aligned} (3) \quad g'(w) &= \frac{dg(w)}{dw} = \left(\frac{dg(w)}{dz} \right) / \left(\frac{dw}{dz} \right) \\ &= \left(\frac{df(z)}{dz} \right) / \left(\frac{dh(z)}{dz} \right) = f'(z)/h'(z). \end{aligned}$$

From the assumption (1) and from (3), we have

$$(4) \quad | \arg (g'(w' + (w'' - w')t) - \beta) | \leq \frac{\alpha\pi}{2} \quad \text{in } E.$$

From the property of integral mean value of (2) and from (4), (see e.g. [3, Lemma 1]), we have

$$| \arg \left(\frac{f(z'') - f(z')}{h(z'') - h(z')} - \beta \right) | \leq \frac{\alpha\pi}{2} \quad \text{in } E.$$

This completes our proof.

Putting $\alpha = 1$ in the main theorem, we have

$$\operatorname{Re} \left(\frac{f'(z)}{h'(z)} - \beta \right) > 0 \quad \text{in } E \quad \implies \quad \operatorname{Re} \left(\frac{f(z'') - f(z')}{h(z'') - h(z')} - \beta \right) > 0 \quad \text{in } E.$$

This is an extended MacGregor's theorem.

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