

ON STARLIKENESS AND CONVEXITY OF
CERTAIN MULTIVALENT FUNCTIONS

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Abstract

The object of the present paper is to determine the radii of starlikeness and convexity of order α of certain analytic multivalent functions with a kind of bounded argument.

1. Introduction

Let p, α, β and r denote $p \in N = \{1, 2, 3, \dots\}$, $0 \leq \alpha < p$, $\beta > 0$ and $0 < r \leq 1$, respectively. Let U_r denote the set $\{z: |z| < r\}$ and let U denote the unit disk U_1 . Next, let A_p denote the class of functions of the form :

$$(1.1) \quad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

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which are analytic in the unit disk U . We denote A_1 by A .

A function $f(z)$ in the class A_p is said to be p -valently starlike of order α in U_r if and only if it satisfies

$$(1.2) \quad \operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \quad (z \in U_r).$$

We denote by $S_p^*(\alpha)_r$ the subclass of the class A_p consisting of all p -valently starlike functions of order α in U_r .

Further, a function $f(z)$ in the class A_p is said to be p -valently convex of order α in U_r if and only if it satisfies

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha \quad (z \in U_r).$$

Also, we denote by $K_p(\alpha)_r$ the subclass of the class A_p consisting of all p -valently convex functions of order α in U_r .

Let M be the class of functions of the form :

$$(1.4) \quad p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

which are analytic in the unit disk U .

A function $p(z)$ in the class M is said to be a member of the class $M(\beta)$ if and only if it satisfies

$$(1.5) \quad |\arg p(z)| < \frac{\pi \beta}{2} \quad (z \in U).$$

Finally, a function $f(z)$ in the class A_p is said to be p -valent strongly close-to-convex of order α and type β in U_r if and

only if there is a function $g(z) \in K_p(\alpha)_r$ such that $\frac{f'(z)}{g'(z)} \in H(\beta)$.

We denote by $C_p(\alpha, H_\beta)_r$ the subclass of the class A_p consisting of all p -valent strongly close-to-convex functions of order α and type β in U_r .

In particular, whenever the numbers p, α, β and r mentioned in technical terms $S_p^*(\alpha)_r, K_p(\alpha)_r$ and $C_p(\alpha, H_\beta)_r$ are equal to 1, 0, 1 and 1, respectively, these numbers are removed from the technical terms. For example,

$$\begin{aligned} S_p^*(\alpha) &= S_p^*(\alpha)_1, & K_p(\alpha) &= K_p(\alpha)_1, & S^*(\alpha)_r &= S_1^*(\alpha)_r, \\ S^*(\alpha) &= S_1^*(\alpha)_1, & K(\alpha)_r &= K_1(\alpha)_r, & C_p(\alpha, H_\beta) &= C_p(\alpha, H_\beta)_1, \\ S^* &= S_1^*(0)_1, & K &= K_1(0)_1, & C &= C_1(0, H_1)_1. \end{aligned}$$

A function $f(z)$ in the classes S^*, K and C is said to be starlike, convex and close-to-convex, respectively.

2. The radii of starlikeness

In order to get our results, we here have to recall Lemma 2.A and prove Lemma 2.1.

Lemma 2.A (Nunokawa and Causey [1]). *Let β be $\beta > 0$. If $p(z) \in H(\beta)$, then*

$$(2.1) \quad \left| \frac{p'(z)}{p(z)} \right| \leq \frac{2\beta}{1 - |z|^2} \quad (z \in U).$$

Lemma 2.1. *Let p and α be $p \in \mathbb{N}$ and $0 \leq \alpha < p$, respectively.*

If $g(z) \in S_p^*(\alpha)$, then

$$(2.2) \quad \operatorname{Re} \left\{ \frac{z g'(z)}{g(z)} \right\} \geq (p - \alpha) \frac{1 - |z|}{1 + |z|} + \alpha \quad (z \in U).$$

Proof. Defining the functions $h(z)$ and $h_0(z)$ by

$$(2.3) \quad h(z) = \frac{z g'(z)}{p g(z)} \quad \text{and} \quad h_0(z) = \left(1 - \frac{\alpha}{p}\right) \frac{1 - z}{1 + z} + \frac{\alpha}{p},$$

respectively, we have

$$h(z), h_0(z) \in H, \quad h(0) = h_0(0) \quad \text{and} \quad \operatorname{Re} h(z) > \frac{\alpha}{p},$$

because of $g(z) \in S_p^*(\alpha)$. Since the function $h_0(z)$ is univalent in U and maps the unit disk U onto $\operatorname{Re} w > \frac{\alpha}{p}$, we obtain

$$\left| h_0^{-1}(h(z)) \right| \leq |z| \quad (z \in U)$$

by using Schwarz's lemma. This inequality shows that the image of the unit disk U by $h(z)$ have to be in the disk whose diameter end points are

$$\left(1 - \frac{\alpha}{p}\right) \frac{1 - |z|}{1 + |z|} + \frac{\alpha}{p} \quad \text{and} \quad \left(1 - \frac{\alpha}{p}\right) \frac{1 + |z|}{1 - |z|} + \frac{\alpha}{p}.$$

This completes the proof of Lemma 2.1.

q. e. d.

Now, we have

Theorem 2.1. Let p, j, α, β and γ be $p \in \mathbb{N}$, $j = 0, 1, 2, \dots, p-1$, $0 \leq \alpha < p-j$, $\beta > 0$ and $0 \leq \gamma < p-j$, respectively. If a function $f(z)$ is in the class A_p and

$$\frac{f^{(j)}(z)}{g^{(j)}(z)} \in H(\beta)$$

for some $g(z) \in A_p$ such that

$$\frac{(p-j)!}{p!} g^{(j)}(z) \in S_{p-j}^*(\alpha),$$

then

$$\frac{(p-j)!}{p!} f^{(j)}(z) \in S_{p-j}^*(\gamma)_r,$$

where

$$(2.4) \quad \begin{cases} r = \frac{p-j-\alpha+\beta-\sqrt{A}}{p-j-2\alpha+\gamma} & \text{if } p-j-2\alpha+\gamma > 0, \\ r = \frac{p-j-\alpha+\beta+\sqrt{A}}{p-j-2\alpha+\gamma} & \text{if } p-j-2\alpha+\gamma < 0, \\ r = \frac{p-j-\alpha}{p-j-\alpha+\beta} & \text{if } p-j-2\alpha+\gamma = 0, \end{cases}$$

and

$$A = (\alpha - \beta - \gamma)^2 + 2\beta(p-j-\gamma).$$

The result is sharp for the function $f(z)$ defined by

$$(2.5) \quad f(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}} \left(\frac{1+z}{1-z} \right)^\beta \quad \text{as } j=0,$$

$$(2.6) \quad f(z) = \int_0^z \int_0^{\xi_j} \dots \int_0^{\xi_2} \frac{p!}{(p-j)!(1-\xi_1)^{2(p-j-\alpha)}} \frac{\xi_1^{p-j}}{\left(\frac{1+\xi_1}{1-\xi_1} \right)^\beta} d\xi_1 \dots d\xi_{j-1} d\xi_j$$

as $j = 1, 2, \dots, p-1,$

at $z = -|z|.$

Proof. Defining the function $p(z)$ by

$$(2.7) \quad p(z) = \frac{f^{(j)}(z)}{g^{(j)}(z)},$$

we have $p(z) \in M(\beta)$. Since

$$(2.8) \quad \frac{p'(z)}{p(z)} = \frac{f^{(j+1)}(z)}{f^{(j)}(z)} - \frac{g^{(j+1)}(z)}{g^{(j)}(z)},$$

we see, using Lemma 2.A, that

$$(2.9) \quad \left| \frac{z f^{(j+1)}(z)}{f^{(j)}(z)} - \frac{z g^{(j+1)}(z)}{g^{(j)}(z)} \right| \leq \frac{2\beta |z|}{1 - |z|^2} \quad (z \in U).$$

The function $\frac{(p-j)!}{p!} g^{(j)}(z) \in S_{p-j}^*(\alpha)$ satisfies

$$(2.10) \quad \operatorname{Re} \left\{ \frac{z g^{(j+1)}(z)}{g^{(j)}(z)} \right\} \geq (p-j-\alpha) \frac{1-|z|}{1+|z|} + \alpha \quad (z \in U),$$

by Lemma 2.1. Therefore, it follows from (2.9) and (2.10) that

$$(2.11) \quad \operatorname{Re} \left\{ \frac{z f^{(j+1)}(z)}{f^{(j)}(z)} \right\} \geq \operatorname{Re} \left\{ \frac{z g^{(j+1)}(z)}{g^{(j)}(z)} \right\} - \frac{2\beta |z|}{1 - |z|^2} \\ \geq (p-j-\alpha) \frac{1-|z|}{1+|z|} + \alpha - \frac{2\beta |z|}{1 - |z|^2} \\ > \gamma, \text{ in } U_r.$$

q. e. d.

We here obtain two corollaries. Putting $j = 0$ in Theorem 2.1, we have Corollary 2.1.

Corollary 2.1. *Let p, α, β and γ be $p \in \mathbb{N}$, $0 \leq \alpha < p$, $\beta > 0$ and $0 \leq \gamma < p$, respectively. If a function $f(z)$ is in the class A_p and*

$$\frac{f(z)}{g(z)} \in M(\beta)$$

for some $g(z) \in S_p^(\alpha)$, then*

$$f(z) \in S_p^*(\gamma)_r,$$

where

$$(2.12) \quad \left\{ \begin{array}{ll} r = \frac{p - \alpha + \beta - \sqrt{B}}{p - 2\alpha + \gamma} & \text{if } p - 2\alpha + \gamma > 0, \\ r = \frac{p - \alpha + \beta + \sqrt{B}}{p - 2\alpha + \gamma} & \text{if } p - 2\alpha + \gamma < 0, \\ r = \frac{p - \alpha}{p - \alpha + \beta} & \text{if } p - 2\alpha + \gamma = 0, \end{array} \right.$$

and

$$B = (\alpha - \beta - \gamma)^2 + 2\beta(p - \gamma).$$

The result is sharp for the function $f(z)$ defined by (2.5) at $z = -|z|$.

Putting $p = 1$ in Corollary 2.1, we have Corollary 2.2.

Corollary 2.2. Let α, β and γ be $0 \leq \alpha < 1, \beta > 0$ and $0 \leq \gamma < 1$, respectively. If a function $f(z)$ is in the class A and

$$\frac{f(z)}{g(z)} \in M(\beta)$$

for some $g(z) \in S^*(\alpha)$, then

$$f(z) \in S^*(\gamma)_r,$$

where

$$(2.13) \quad \left\{ \begin{array}{ll} r = \frac{1 - \alpha + \beta - \sqrt{C}}{1 - 2\alpha + \gamma} & \text{if } 1 - 2\alpha + \gamma > 0, \\ r = \frac{1 - \alpha + \beta + \sqrt{C}}{1 - 2\alpha + \gamma} & \text{if } 1 - 2\alpha + \gamma < 0, \\ r = \frac{1 - \alpha}{1 - \alpha + \beta} & \text{if } 1 - 2\alpha + \gamma = 0, \end{array} \right.$$

and

$$C = (\alpha - \beta - \gamma)^2 + 2\beta(1 - \gamma).$$

The result is sharp for the function $f(z)$ defined by

$$(2.14) \quad f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \left(\frac{1+z}{1-z} \right)^\beta$$

at $z = -|z|$.

Remark 2.1. Taking $\alpha = 0$ in Corollary 2.2, we have the corresponding result due to Yaguchi, Obradović, Nunokawa and Owa [3]. Furthermore, taking $\gamma = 0$ and $\beta = 2$ in Corollary 2.2, we have the corresponding result due to Yaguchi and Nunokawa [2].

3. The radius of convexity

Noting that $f(z) \in K_p(\alpha)$ if and only if $zf'(z) \in S_p^*(\alpha)$, we get the following result with the aid of Theorem 2.1.

Theorem 3.1. Let p, j, α, β and γ be in the same conditions as in Theorem 2.1. If a function $f(z)$ is in the class A_p and

$$\frac{f^{(j+1)}(z)}{g^{(j+1)}(z)} \in M(\beta)$$

for some $g(z) \in A_p$ such that

$$\frac{(p-j)!}{p!} g^{(j)}(z) \in K_{p-j}(\alpha),$$

then

$$\frac{(p-j)!}{p!} f^{(j)}(z) \in K_{p-j}(\gamma)_r,$$

where r is given by (2.4). The result is sharp for the function $f(z)$ defined by

$$(3.1) \quad f(z) = \int_0^z \int_0^{\xi_j} \cdots \int_0^{\xi_1} \frac{p! \xi_0^{p-j-1}}{(p-j-1)!(1-\xi_0)^{2(p-j-\alpha)}} \left(\frac{1+\xi_0}{1-\xi_0} \right)^\beta d\xi_0 \cdots d\xi_{j-1} d\xi_j,$$

at $z = -|z|$.

Proof. Defining the functions $F(z)$ and $G(z)$ by

$$F(z) = \frac{(p-j-1)!}{p!} z f^{(j+1)}(z)$$

and

$$G(z) = \frac{(p-j-1)!}{p!} z g^{(j+1)}(z),$$

respectively. We have

$$F(z) \in A_{p-j}, \quad G(z) \in S_{p-j}^*(\alpha) \quad \text{and} \quad \frac{F(z)}{G(z)} \in H(\beta).$$

By Corollary 2.1, we obtain that $F(z) \in S_{p-j}^*(\gamma)_r$, where r is given

by (2.2). Therefore $\frac{(p-j)!}{p!} f^{(j)}(z)$ is $(p-j)$ -valently convex of

order γ in U_r .

q. e. d.

Putting $j = 0$ in Theorem 3.1, we have Corollary 3.1.

Corollary 3.1. Let p, α, β and γ be in the same conditions as in Corollary 2.1. If a function $f(z)$ in A_p is in the class

$C_p(\alpha, M_\beta)$, then $f(z) \in K_p(\gamma)_r$, where r is given by (2.12). The result

is sharp for the function $f(z)$ defined by

$$(3.2) \quad f(z) = \int_0^z \frac{p \xi^{p-1}}{(1-\xi)^{2(p-\alpha)}} \left(\frac{1+\xi}{1-\xi} \right)^\beta d\xi,$$

at $z = -|z|$.

Putting $p = 1$ in Corollary 3.1, we have Corollary 3.2.

Corollary 3.2. Let α , β and γ be in the same conditions as in Corollary 2.2. If a function $f(z)$ in the class A is in the class $C(\alpha, M_\beta)$, then $f(z) \in K(\gamma)_r$, where r is given by (2.13). The result is sharp for the function $f(z)$ defined by

$$(3.3) \quad f(z) = \int_0^z \frac{1}{(1-\xi)^{2(1-\alpha)}} \left(\frac{1+\xi}{1-\xi} \right)^\beta d\xi$$

at $z = -|z|$.

Putting $\beta = 1$ in Corollary 3.2, we have Corollary 3.3.

Corollary 3.3. If a function $f(z)$ in the class A is close-to-convex of order α ($0 \leq \alpha < 1$), then $f(z)$ is convex of order γ in U_r , where

$$(3.4) \quad \begin{cases} r = \frac{2 - \alpha - \sqrt{D}}{1 - 2\alpha + \gamma} & \text{if } 1 - 2\alpha + \gamma > 0, \\ r = \frac{2 - \alpha + \sqrt{D}}{1 - 2\alpha + \gamma} & \text{if } 1 - 2\alpha + \gamma < 0, \\ r = \frac{1 - \alpha}{2 - \alpha} & \text{if } 1 - 2\alpha + \gamma = 0, \end{cases}$$

and

$$D = (\alpha - \gamma)^2 - 2\alpha + 3.$$

The result is sharp for the function $f(z)$ defined by

$$(3.5) \quad f(z) = \frac{1}{(2\alpha - 1)(1 - \alpha)} \left(\frac{\alpha - (1 - \alpha)z}{(1 - z)^{2(1-\alpha)}} - \alpha \right) \quad \left(\alpha \neq \frac{1}{2} \right)$$

$$(3.6) \quad f(z) = \frac{2z}{1 - z} + \log(1 - z) \quad \left(\alpha = \frac{1}{2} \right)$$

at $z = -|z|$.

Putting $\alpha = 0$ in Corollary 3.3, we have Corollary 3.4.

Corollary 3.4. *If a function $f(z)$ in the class A is close-to-convex, then $f(z)$ is convex of order γ in U_r ,*

$$\text{where } r = \frac{2 - \sqrt{\gamma^2 + 3}}{1 + \gamma}.$$

The result is sharp for the Koebe function

$$f(z) = \frac{z}{(1 - z)^2}$$

at $z = -|z|$.

References

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