数理解析研究所講究録
第 714 巻 1990 年 149-159

On certain class of analytic functions with negative coefficients

日大薬 関根忠行 (Tadayuki Sekine)

1. Introduction and Definition.

In [3] we introduced the class $A(\alpha)$ and the subclass $A(\alpha,\beta)$ of $A(\alpha)$ as follows.

Let $A(\alpha)$ denote the class of analytic functions $f(z)$ of the form

(1.1) $$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (e^{i\alpha}a_n \geq 0,\ |\alpha| < \frac{\pi}{2})$$

in the unit disk $U = \{z: |z| < 1\}$. Also let $A(\alpha,\beta)$ denote the subclass of $A(\alpha)$ consisting of functions which satisfy the inequality

(1.2) $$\mathrm{Re}\{e^{i\alpha}f'(z)\} > \beta \qquad (0 \leq \beta < \cos\alpha).$$

Class of this type for $\alpha = 0$ was investigated by Sarangi and Uralegaddi [1].

For the subclass $A(\alpha,\beta)$ of $A(\alpha)$, we obtained the following result.

<u>Lemma([3; Theorem 1]).</u> A function $f(z)$ is in $A(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty} ne^{i\alpha}a_n \leq \cos\alpha - \beta. \tag{1.3}$$

The result is sharp.

Using the lemma, we [3] determined distortion inequalities and the radius of convexity and starlikeness of functions in the class $A(\alpha,\beta)$. Further we showed a result for the quasi-Hadamard products.

In this report we introduce a subclass $R(\alpha,\beta)$ of the class $A(\alpha)$ and a subclass $A_\gamma(\alpha,\beta)$ which means a interpolate of two subclass $A(\alpha,\beta)$ and $R(\alpha,\beta)$. Some results [3] on the subclass $A(\alpha,\beta)$ are generalized to the case of subclass $A_\gamma(\alpha,\beta)$.

Let $R(\alpha,\beta)$ denote the subclass of $A(\alpha)$ consisting of function which satisfy the inequality

$$\mathrm{Re}\{e^{i\alpha}\frac{f(z)}{z}\} > \beta \qquad (0 \leq \beta < \cos\alpha). \tag{1.5}$$

Class of this type for $\alpha = 0$ was studied by Sarangi and Uralegaddi [1].

By using the same manner as the proof of Lemma, we easily obtain the following theorem.

<u>Theorem 1.</u> A function $f(z)$ is in $R(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty} e^{i\alpha}a_n \leq \cos\alpha - \beta. \tag{1.5}$$

The result is sharp for the function

$$f(z) = z - (\cos\alpha - \beta)e^{-i\alpha}z^n \quad (n \geq 2). \tag{1.7}$$

Now we introduce a subclass $A_\gamma(\alpha,\beta)$ of the class $A(\alpha)$. We say that a function $f(z)$ belongs to the class $A_\gamma(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty} (\gamma n + 1 - \gamma)e^{i\alpha}a_n \leq \cos\alpha - \beta \quad (0 \leq \gamma \leq 1). \tag{1.8}$$

Evidently, $A_0(\alpha,\beta) = R(\alpha,\beta)$ and $A_1(\alpha,\beta) = A(\alpha,\beta)$.

2. Distortion inequalities and the radius of convexity and starlikeness.

<u>Theorem 2.</u> If function $f(z)$ is in $A_\gamma(\alpha,\beta)$ $(0 \leq \gamma \leq 1)$, then

$$\text{(i)} \quad |z| - \frac{\cos\alpha - \beta}{1 + \gamma}|z|^2 \leq |f(z)| \leq |z| + \frac{\cos\alpha - \beta}{1 + \gamma}|z|^2, \tag{2.1}$$

$$\text{(ii)} \quad 1 - \frac{2(\cos\alpha - \beta)}{1 + \gamma}|z| \leq |f'(z)| \leq 1 + \frac{2(\cos\alpha - \beta)}{1 + \gamma}|z| \quad (\gamma \neq 0). \tag{2.2}$$

The results are sharp for the function

$$f(z) = z - \frac{\cos\alpha - \beta}{1 + \gamma}\, e^{-i\alpha}z^2. \tag{2.3}$$

Proof. (i) We have

(2.4) $$|f(z)| \le |z| + |z|^2 \sum_{n=2}^{\infty} |a_n|$$

By coefficients inequalities (1.8), it follows that

$$(1 + \gamma) \sum_{n=2}^{\infty} e^{i\alpha} a_n \le \sum_{n=2}^{\infty} (\gamma n + 1 - \gamma) e^{i\alpha} a_n \le \cos\alpha - \beta$$

that is, that

(2.5) $$\sum_{n=2}^{\infty} |a_n| \le \frac{\cos\alpha - \beta}{1 + \gamma}.$$

Substituting (2.5) into (2.4) we obtain the right-hand side inequality of (i). On the other hand, we have

(2.6) $$|f(z)| \ge |z| - |z|^2 \sum_{n=2}^{\infty} |a_n|$$

$$\ge |z| - |z|^2 \frac{\cos\alpha - \beta}{\gamma + 1}$$

(ii) $$1 - |z| \sum_{n=2}^{\infty} n|a_n| \le |f'(z)| \le 1 + |z| \sum_{n=2}^{\infty} n|a_n|$$

By (1.8) we see that

(2.7) $$\sum_{n=2}^{\infty} n|a_n| \le \frac{2(\cos\alpha - \beta)}{1 + \gamma} \quad (\gamma \ne 0).$$

Thus assertion follows.

If we put $\gamma = 1$ in Theorem 1, we shall obtain the same result given by Sekine [3;Theorem 2].

Theorem 3. If $f(z)$ is in $A_\gamma(\alpha,\beta)$, then $f(z)$ is convex of order $\delta (0 \le \delta < 1)$ in the disk

(2.8) $$|z| < r_1 = \inf_{n\ge 2}\left\{ \frac{(1-\delta)(\gamma n + 1 - \gamma)}{\eta(n-\delta)(\cos\alpha - \beta)} \right\}^{\frac{1}{n-1}} \quad (n \ge 2).$$

The result is sharp for the function

(2.9) $$f(z) = z - \frac{\cos\alpha - \beta}{(\gamma n + 1 - \gamma)} e^{-i\alpha} z^n \quad (n \ge 2).$$

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| > 1 - \delta \quad \text{for } |z| < r_1.$$

We have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left| \frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} na_n z^{n-1}} \right|$$

$$\le \frac{\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} |a_n||z|^{n-1}}.$$

Hence $\left|\frac{zf''(z)}{f'(z)}\right| < 1 - \delta$ if

(2.10) $$\sum_{n=2}^{\infty} \frac{n(n - \delta)|a_n||z|^{n-1}}{1 - \delta} < 1.$$

By (1.8) we see that

(2.11) $$\sum_{n=2}^{\infty} \frac{(\gamma n + 1 - \gamma)|a_n|}{\cos\alpha - \beta} \leq 1.$$

Hence (2.8) is satisfied if

$$\frac{n(n - \delta)|a_n||z|^{n-1}}{1 - \delta} < \frac{(\gamma n + 1 - \gamma)|a_n|}{\cos\alpha - \beta} \quad (n \geq 2).$$

Solving this for $|z|$, we get

(2.12) $$|z| < \left\{ \frac{(1 - \delta)(\gamma n + 1 - \gamma)}{n(n - \delta)(\cos\alpha - \beta)} \right\}^{\frac{1}{n-1}} \quad (n \geq 2).$$

Writing

$$r_1 = \inf_{n\geq 2} \left\{ \frac{(1 - \delta)(\gamma n + 1 - \gamma)}{n(n - \delta)(\cos\alpha - \beta)} \right\}^{\frac{1}{n-1}} \quad (n \geq 2),$$

in (2.12), the result follows.

If we put $\gamma = 1$ in Theorem 3, we shall obtain the same result [3; Theorem 3].

Theorem 4. If $f(z)$ is in $A_\gamma(\alpha,\beta)$, then $f(z)$ is starlike of order $\delta (0 \leq \delta < 1)$ in the disk

$$(2.13)\qquad |z| < r_2 = \inf_{n\geq 2} \left\{ \frac{n(1-\delta)(\gamma n + 1 - \gamma)}{(n-\delta)(\cos\alpha - \beta)} \right\}^{\frac{1}{n-1}}.$$

The result is sharp for the function (2.9).

Proof. It sufficies to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta \quad \text{for } |z| < r_2.$$

We have

$$(2.14)\qquad \left|\frac{zf'(z)}{f(z)} - 1\right| = \left| \frac{-\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} \right|$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty}|a_n||z|^{n-1}}.$$

Hence $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta$ if

$$(2.15)\qquad \sum_{n=2}^{\infty} \frac{(n-\delta)|a_n||z|^{n-1}}{1-\delta} < 1.$$

The remaining part of the proof is similar to that of Theorem 3.

If we put $\gamma = 1$ in Theorem 4, we shall obtain the same result [3; Theorem 4].

3. Quaqsi-Hadamard product

Let the functions in the class $A_\gamma(\alpha,\beta)$ be of the form

(3.1) $$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (e^{i\alpha} a_n \geq 0,\ |\alpha| < \frac{\pi}{2}),$$

(3.2) $$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (e^{i\alpha} b_n \geq 0,\ |\alpha| < \frac{\pi}{2})$$

and define the quasi-Hadamard product $(f*g)(z)$ of the functions $f(z)$ and $g(z)$ by

(3.3) $$(f*g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n.$$

<u>Theorem 6.</u> If $f(z)$ and $g(z)$ are in $A_\gamma(\alpha_1,\beta_1)$ and $A_\mu(\alpha_2,\beta_2)$ respectively, then $(f*g)(z)$ is in the class $A_\nu(\alpha_1+\alpha_2,\lambda)$ excepting in the case of $\gamma = \mu = 0$ and $\nu = 1$, where

(3.4) $$\lambda = \cos(\alpha_1 + \alpha_2) - \frac{(\nu + 1)(\cos\alpha_1 - \beta_1)(\cos\alpha_2 - \beta_2)}{(\gamma + 1)(\mu + 1)}.$$

Proof. By coefficient inequality (1.8), we have

(3.5) $$\sum_{n=2}^{\infty} \frac{\gamma n + 1 - \gamma}{\cos\alpha_1 - \beta_1} e^{i\alpha_1} a_n \leq 1$$

and

(3.6) $$\sum_{n=2}^{\infty} \frac{\mu n + 1 - \mu}{\cos\alpha_2 - \beta_2} e^{i\alpha_2} b_n \leq 1.$$

We need to find the largest λ such that

(3.7) $$\sum_{n=2}^{\infty} \frac{(\nu n + 1 - \nu) e^{i(\alpha_1 + \alpha_2)} a_n b_n}{\cos(\alpha_1 + \alpha_2) - \lambda} \leq 1.$$

Applying Cauchy-Schwarz inequality to (3.4) and (3.5), we have

(3.8) $$\sum_{n=2}^{\infty} \sqrt{\frac{(\gamma n + 1 - \gamma) e^{i\alpha_1} a_n}{\cos\alpha_1 - \beta_1}} \sqrt{\frac{(\mu n + 1 - \mu) e^{i\alpha_2} b_n}{\cos\alpha_2 - \beta_2}} \leq 1.$$

Then we want show that

(3.9) $$\frac{(\nu n + 1 - \nu) e^{i(\alpha_1 + \alpha_2)} a_n b_n}{\cos(\alpha_1 + \alpha_2) - \lambda}$$

$$\leq \sqrt{\frac{(\gamma n + 1 - \gamma) e^{i\alpha_1} a_n}{\cos\alpha_1 - \beta_1}} \sqrt{\frac{(\mu n + 1 - \mu) e^{i\alpha_2} b_n}{\cos\alpha_2 - \beta_2}} \quad (n \geq 2)$$

that is , that

(3.10) $$\sqrt{e^{i\alpha_1} a_n} \sqrt{e^{i\alpha_2} b_n}$$

$$\leq \frac{\sqrt{\gamma n + 1 - \gamma}\,\sqrt{\mu n + 1 - \mu}\,\{\cos(\alpha_1 + \alpha_2) - \lambda\}}{(\nu n + 1 - \nu)\sqrt{\cos\alpha_1 - \beta_1}\,\sqrt{\cos\alpha_2 - \beta_2}} \quad (n \geq 2).$$

Since we have

$$\sqrt{e^{i\alpha_1} a_n}\,\sqrt{e^{i\alpha_2} b_n} \leq \frac{\sqrt{\cos\alpha_1 - \beta_1}\,\sqrt{\cos\alpha_2 - \beta_2}}{\sqrt{\gamma n + 1 - \gamma}\,\sqrt{\mu n + 1 - \mu}}$$

by (3.8), if

$$\frac{\sqrt{\cos\alpha_1 - \beta_1}\,\sqrt{\cos\alpha_2 - \beta_2}}{\sqrt{\gamma n + 1 - \gamma}\,\sqrt{\mu n + 1 - \mu}}$$

$$\leq \frac{\sqrt{\gamma n + 1 - \gamma}\,\sqrt{\mu n + 1 - \mu}\,\{\cos(\alpha_1 + \alpha_2) - \lambda\}}{(\nu n + 1 - \nu)\sqrt{\cos\alpha_1 - \beta_1}\,\sqrt{\cos\alpha_2 - \beta_2}}$$

(3.7) is true. Solving the above inequality for λ, we obtain

$$(3.11)\quad \lambda \leq \cos(\alpha_1 + \alpha_2) - \frac{(\cos\alpha_1 - \beta_1)(\cos\alpha_2 - \beta_2)(\nu n + 1 - \nu)}{(\gamma n + 1 - \gamma)(\mu n + 1 - \mu)}.$$

We note that the right-hand side of (3.11) is an increasing function of $n(n \geq 2)$, then writing $n = 2$ in (3.11) we conclude

$$(3.12)\quad \lambda \leq \cos(\alpha_1 + \alpha_2) - \frac{(\nu + 1)(\cos\alpha_1 - \beta_1)(\cos\alpha_2 - \beta_2)}{(\gamma + 1)(\mu + 1)}.$$

Letting $\gamma = \mu = \nu = 1$, $\alpha_1 = \alpha_2$ and $\beta_2 = \beta_2$ in Theorem 6, we have the same result [3; Theorem 5].

By Theorem 6, we easily obtain the following corollary.

Corollary 1. If functions $f_i(z)(i = 1,2,3,\ldots,p)$ are in $A_\gamma(\alpha,\beta)$, then $(f_1*f_2*f_3*\ldots*f_p)(z)$ is in the class $A_\gamma(p\alpha,\lambda)$, where

$$\lambda = \cos p\alpha - \frac{(\cos\alpha - \beta)^p}{(\gamma + 1)^{p-1}} \quad (p \geq 2). \tag{3.13}$$

References

[1] S.M.Sarangi and B.A. Uralegaddi, The radius of convexity and starlikeness for certain classes of analytic functions with negative coefficients I, Rend, Accd. Naz. Lincei, 65(1978) 38-42.

[2] A.Shild and H.Silverman, Convolutions of univalent functions with negative coefficients, Ann. Univ. Mariae Curie-Sklodowska (1975)99-106.

[3] T.Sekine, On generalyzed class of analytic functions with negative coefficients, submitted to Mathematica Japonica.