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Lorentz structures and Killing vector fields on manifolds

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This is a brief summary on the structure of Lorentz manifolds of constant curvature. Our results are stated in Sections 7, 8 and 9 without proof and the detail will be published elsewhere.

We apologize not to have the space to state the results on CR-structures on manifolds. We refer to [K-T] for the result. A Lorentz manifold M of dimension n (≥ 1) is a smooth manifold together with a Lorentz metric g. A Lorentz metric g on M is a smooth field $\{g_x\}_{x\in M}$ of nondegenerate symmetric bilinear forms g_x of type $(1, n-1)$ on the tangent space T_xM. Nemely let $\mathbf{R}^{1,n-1}$ denote the real vector space of dimension n equipped with the bilinear form

$$Q(x,y) = -x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

A nondegenarate symmetric bilinear form g_x is of type $(1, n-1)$ if the pair (T_xM, g_x) is isometric to $(\mathbf{R}^{1,n-1}, Q)$. (See [Wo],[O'Ne].)

In general a pseudo-Riemannian manifold is a smooth manifold together with a pseudo-Riemannian metric(an indefinite metric). It is the fundamental result in Riemannian Geometry that a pseudo-Riemannian manifold has a unique connection (Levi-Cività connection) on its frame bundle. And thus geodesics, curvature, completenes etc. will refer to the Levi-Cività connection). In paraticular the sectional curvature will be defined. Also a pseudo-Riemannian manifold M is complete if the Levi-Cività connection is complete, i.e., every geodesic segment $[0,1] \to M$ can be extended to a full geodesic. In comparison to Riemannian manifolds, not every smooth manifold admits a pseudo-Riemannian metric.

It is notorious that compactness does not necessarily imply completeness. It is of interest to examine this feature in Lorentz Geometry. In this paper we shall concern this problem for Lorentz manifolds of constant curvature admitting Killing vector fields. As is noted above, the sectional curvature is defined on Lorentz manifolds. Then there is a class of Lorentz manifolds of constant curvature among all Lorentz manifolds.

1. Lorentz Causal Character.

Let M be a Lorentz manifold with metric g. A tangent vector v to M falls into the following type:

$$\begin{array}{llll} timelike & if & g(v,v) < 0, & \\ lightlike & if & g(v,v) = 0, & and \\ spacelike & if & g(v,v) > 0. & \end{array}$$

A curve γ in M is timelike if all of the velocity vectors $\gamma'(t)$ are timelike; similarly for lightlike and spacelike. We remark that an arbitrary curve need not have one of these causal characters but a geodesic does, i.e., $g(\gamma\prime(t), \gamma\prime(t))$ is constant. This is because $\gamma\prime$ is parallel and parallel translation preserves causality.

2. Exitence of Lorentz metric

As to the existence of Lorentz metrics on smooth manifolds, we notice that *M admits a Lorentz metric if and only if there exists a nonzero vector field on M.* (See [O'Ne, p.149].) And so if either M is noncompact or M is compact and has euler characteristic $\chi(M) = 0$, then M admits a Lorentz metric. We have the following result.

LEMMA 1 (CF.[O'NE]). *If M admits a nonzero vector field V then M admits a Lorentz metric such that V is timelike.*

For this, let g be a Riemannian metric on M so that V is a unit vector field. Define a new metric by setting

$$h(X,Y) = g(X,Y) - 2g(V,X)\cdot g(V,Y).$$

Remark 1. A Lorentz manifold is called time-orientable if it admits a timelike vector field.

3. Isomery of Lorentz Manifolds.

Let Iso(M) denote the group of all isometries of a Lorentz manifold M onto itself. It is known that Iso(M) is a (finite) dimensional Lie group. Let X be a complete vector field on M. Then X generates a one parameter group $\{\phi_t\}$ of diffeomorphisms of M. A vector field X is Killing if each ϕ_t is an isometry, i.e., $\{\phi_t\} \in$ Iso(M). When M is a complete Lorentz manifold, it follows that the Lie algebra of Iso(M) is isomorphic to the Lie algebra $i(M)$ consisting of all Killing vector fields.

It is a famous result that if M is a *Riemannian* manifold then Iso(M) acts properly on M. In particular the stabilizer at any point of M is compact. In addition Iso(M) is compact if M is compact. However, in *Lorentz geometry* it is noted that Iso(M) of a Lorentz manifold M is not necessarily compact even if M is compact. (See [D'Am] for a related work.) Moreover Iso(M) need not act properly and hence its stabilizer fails to be compact. Therefore, the necessary condition that a group Γ is discrete in Iso(M) is not a sufficient condition for Γ to act properly discontinuously on a Lorentz manifold M. This fact makes difficult to understanding the topology of Lorentz manifolds (cf. [Ku],[K-R],[Wo]).

4. Models for Complete Lorentz Manifold.

Consider the following quadrics;

$$\begin{aligned}
\mathbf{S}^{1,n} &= \{\ p = (x_1, y_1, \cdots, y_{n+1}) \in \mathbf{R}^{1,n+1} |\ -x_1^2 + y_1^2 + \cdots + y_{n+1}^2 = 1\}, \\
\mathbf{H}^{1,n} &= \{\ p = (x_1, x_2, y_1, \cdots, y_n) \in \mathbf{R}^{2,n} |\ -x_1^2 - x_2^2 + y_1^2 + \cdots + y_n^2 = -1\}
\end{aligned}$$

Note that $\mathbf{S}^{1,n} \approx \mathbf{R}^1 \times \mathbf{S}^n$, $\mathbf{H}^{1,n} \approx \mathbf{S}^1 \times \mathbf{R}^n$. It follows that $\mathbf{S}^{1,n}$ and $\mathbf{H}^{1,n}$ are complete Lorentz n+1 dimensional manifolds of constant curvature 1 and -1 respectively. The groups $O(1, n+1)$ and $O(2, n)$ are the orthogonal subgroups of GL($n+2, \mathbf{R}$) which preserve the quadratic forms

$$Q^{+}(x_1, y_1, \cdots, y_{n+1}) = -x_1^2 + y_1^2 + \cdots + y_{n+1}^2 ;$$
$$Q^{-}(x_1, x_2, y_1 \cdots, y_n) = -x_1^2 - x_2^2 + y_1^2 + \cdots + y_n^2.$$

Then it follows that $O(1, n+1) = \mathrm{Iso}(\mathbf{S}^{1,n})$ and $O(2,n) = \mathrm{Iso}(\mathbf{H}^{1,n})$.

Let $\tilde{\mathbf{S}}^{1,n}$ be the universal covering space of $\mathbf{S}^{1,n}$. Denote by $O(1, n+1)^{\sim}$ the corresponding group of $O(1, n+1)$ to $\tilde{\mathbf{S}}^{1,n}$. Similarly let $O(2,n)^{\sim}$ be the corresponding group of $O(2,n)$ to the universal covering space $\tilde{\mathbf{H}}^{1,n}$. It is obvious that they are the full groups of isometries of $\tilde{\mathbf{S}}^{1,n}$ and $\tilde{\mathbf{H}}^{1,n}$ respectively. Note that the above vector space $\mathbf{R}^{1,n}$ is a complete connected simply connected Lorentz manifold of zero curvature. The Lorentz metric is obtained by euclidean parallel translation of the above form Q (cf. [Wo],[O'Ne]). We simply denote it by $\mathbf{R}^{n+1}$. The group of isometries of $\mathbf{R}^{n+1}$ is isomorphic to the semidirect product $\mathbf{R}^{n+1} \rtimes O(1,n)$.

We have models for complete connected simply connected Lorentz n+1 dimensional manifolds of constant curvature k and with groups of isometries;

$$\begin{array}{ll} (O(1,n+1)^{\sim}, \tilde{\mathbf{S}}^{1,n}) & if\ k = 1, \\ (\mathbf{R}^{n+1} \rtimes O(1,n), \mathbf{R}^{n+1}) & if\ k = 0,\ and \\ (O(2,n)^{\sim}, \tilde{\mathbf{H}}^{1,n}) & if\ k = -1. \end{array}$$

5. Lorentz Structure.

By (G, X) we shall mean one of the above geometries. We denote that *a Lorentz spherical structure (resp. Lorentz flat structure, and Lorentz hyperbolic structure) on an n+1 dimensional manifold M is a geometric structure modelled on X whose coordinate changes lie in G* where (G, X) represents one of the above for k = 1, 0 and -1 respectively.

A Lorentz spherical (resp. flat and hyperbolic) manifold M is a smooth manifold equipped with a Lorentz spherical (resp. flat and hyperbolic) structure. By the usual monodromy argument if we are given a Lorentz manifold M there exist an immersion $\mathrm{dev} : \tilde{M} \to X$ which preserves the Lorentz structure and a homomorphism $\rho : \pi_1(M) \to$

G where $\tilde{M}$ is the universal covering space. The developing pair (ρ, dev) is uniquely determined up to conjugation. Moreover ρ extends to a homomorphism of $\mathrm{Iso}(\tilde{M})$ into G. Therefore we have the developing pair

$$(\rho, \mathrm{dev}) : (\mathrm{Iso}(\tilde{M}), \tilde{M}) \longrightarrow (G, X)$$

such that $\pi_1(M) \subset \mathrm{Iso}(\tilde{M})$.

By a Lorentz space form we shall mean a complete Lorentz manifold of constant curvature. It is noted that a Lorentz manifold is complete if the developing map is a covering map. The following is the Lorentz space form problem :

THEOREM 1 (KILLING, HOPF). *Let M be a Lorentz space form of dimension n+1 (n$\geq$ 1). Then M is isometric up to a scalar multiple to a quotient*

$\tilde{\mathbf{S}}^{1,n}/\Gamma$	*where* $\Gamma \subset O(1, n+1)^{\sim}$	*if* $k = 1$.
$\mathbf{R}^{n+1}/\Gamma$	*where* $\Gamma \subset \mathbf{R}^{n+1} \rtimes O(1, n)$	*if* $k = 0$.
$\tilde{\mathbf{H}}^{1,n}/\Gamma$	*where* $\Gamma \subset O(2, n)^{\sim}$	*if* $k = $ -1.

Here Γ acts properly discontinuously and freely.

6. Review of Lorentz Space Forms and Current Development.

We recall that (cf. [Wo])

THEOREM 2. *If M is a Lorentz space form $\tilde{\mathbf{S}}^{1,n}/\Gamma$ then Γ is a finite subgroup of $O(1) \times O(n+1)$ up to conjugacy.*

Hence the classification goes back to that of Riemannian spherical space forms. In particular there exist no compact Lorentz spherical space forms.

It has been proved in [G-K] that

THEOREM 3. *If M is a compact Lorentz flat space form $\mathbf{R}^{n+1}/\Gamma$ then Γ is a virtually polycyclic. Further M is diffeomorphic to an infrasolvmanifold.*

See [To] for a generalization. The situation of the noncompact case is quite different from the compact case. Margulis ([M]) gave an interesting example;

THEOREM 4. *There exists a noncompact Lorentz flat space form of dimension three whose fundamental group is isomorphic to a free group of rank two.*

See [D-G] for a generalization. In particular this gives an example of noncompact Lorentz flat space form with nonzero euler characteristic. (Note that every compact complete affine flat manifold has vanishing euler characteristic.) To our later use, we quote the following result ([Ca]) which is concerned with the Markus conjecture.

THEOREM 5. *If M is a compact Lorentz flat manifold then the developing map is a covering map, i.e., M is complete.*

Kulkarni and Raymond ([K-R]) have made a progress on compact Lorentz hyperbolic space forms of dimension three.

THEOREM 6. *Let M be a compact Lorentz hyperbolic space form of dimension three. Then M is finitely covered by a circle bundle with nonzero euler class over a closed surface of genus $g \geqq 2$*

The first author gave an nontrivial example of compact Lorentz hyperbolic space forms. They are called standard space forms (cf. [K-R],[Ku]) and are homeomorphic to Seifert fiber spaces over hyperbolic orbifolds. More precisely, a three dimensional standard space form is a compact Lorentz hyperbolic space form $\tilde{\mathbf{H}}^{1,2}/\Gamma$ whose fundamental group Γ sits in the subgroup $\mathbf{R} \underset{\mathbf{Z}}{\times} \widetilde{\mathrm{PSL}_2\mathbf{R}}$ of $O(2,2)^{\sim}$. In other words, a standard space form is a compact Lorentz hyperbolic space form which admits a timelike Killing vector field induced by a circle action. We remark that a compact Lorentz hyperbolic space form is not always a standard one. In fact there is a deformation of Lorentz hyperbolic structure starting at a standard space form. This was obtained by Goldman ([G]).

THEOREM 7. *There exists a nonstandard Lorentz space form of dimension three.*

In summary we obtain the following.

COROLLARY 1. *Let M be a compact Lorentz space form of dimension three.*

None	*if*	$k = 1$.
an infrasolvmanifold $\mathbf{R}^3/\Gamma$	*if*	$k = 0$.
a Seifert fiber space $\tilde{\mathbf{H}}^{1,2}/\Gamma$	*if*	$k = -1$.

Here Γ acts properly discontinuously and freely.

We have examined connected subgroups of the isometry groups of connected simply connected Lorentz space forms, i.e.,

(1) Connected Subgroups of $O(1, n+1)$
(2) Connected Subgroups of $\mathbf{R}^{n+1} \rtimes O(1, n)$
(3) Connected Subgroups of $O(2, n+1)^{\sim}$

7. Compact Lorentz Spherical Structure.

THEOREM 8. *There exist no timelike or lightlike Killing vector fields on Lorentz spherical manifolds of arbitrary dimensions.*

THEOREM 9. *There exist no compact Lorentz spherical 3-manifold admitting one parameter group of spacelike transformations.*

8. Compact Lorentz Flat Structure.

We notice that every infrasolvmanifold of dimension three supports a complete Lorentz flat structure.

THEOREM 10. *If a compact Lorentz flat 3-manifold admits a one parameter group of spacelike transformations then it is a euclidean space form.*

THEOREM 11. *If a compact Lorentz flat (n+1)-manifold admits a one parameter group of timelike parallel transformations then it is a euclidean space form.*

COROLLARY 2. *A compact Lorentz flat 3-manifold admitting a one parameter group of timelike transformations is a euclidean spaceform.*

THEOREM 12. *If a compact Lorentz flat 3-manifold admits a one parameter group of lightlike transformations then it is an infranilmanifold.*

9. Compact Lorentz Hyperbolic Structure.

We recall examples of compact Lorentz hyperbolic manifolds from [Ku]. They are called standard space forms due to Kulkarni.

THEOREM 13. *If a compact Lorentz hyperbolic manifold admits a one parameter group of timelike transformations then it is complete and some finite covering is diffeomorphic to a circle bundle over a negatively curved manifold.*

THEOREM 14. *Let M be a compact Lorentz hyperbolic manifold which admits a one parameter group $\mathbf{H}$ of Lorentz transformations and $(\rho, \mathrm{dev}) : (\pi, \tilde{\mathbf{H}}, \tilde{M}) \to (\Gamma, \tilde{\mathbf{G}}, \tilde{\mathbf{H}}^{1,n})$ be the developing pair. Let $1 \to \mathcal{Z} \to O(2,n)^{\sim} \xrightarrow{P} O(2,n) \to 1$ be the projection. Put $P(\tilde{\mathbf{G}}) = \mathbf{G}$. If G is compact then we have*

(1) *$\mathbf{H}$ is timelike.*

(2) *The dimension of M is odd and M is a standard space form, i.e., $M^{2n+1} \approx U(n)^{\sim} \backslash U(1,n)^{\sim} / \Gamma$*

THEOREM 15. *If a compact Lorentz hyperbolic 3-manifold admits a timelike Killing vector field then it is a standard space form.*

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