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The third order asymptotic admissibility of estimators in one parameter regular case

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$X_1, X_2, \cdots X_n$ は i. i. d. で共通の確率密度関数 $f(x, \theta_0)$, $\theta_0 \in \Theta \subset R$ を持つものとする。$f(x, \theta_0)$ は次の条件を満たす.

A 1. Θ は R の開集合である.

A 2. $f(x, \theta)$ は (x, θ) に関して可測である.

A 3. 各 x に対して, $f(x, \theta)$ は θ に関して 5 回連続偏微分可能で, その微分係数は θ に関して連続である.

A 4. $\{f(x, \theta),\ \theta \in \Theta\}$ に対応する測度は互いに絶対連続である.

A 5. すべての $\theta \in \Theta$ に対して,

$$E_\theta \left| \log f(X, \theta) \right| < \infty$$

$$0 < I_{1,1}(\theta) = E_\theta \left(\frac{\partial}{\partial\theta} \log f(X, \theta) \right)^2 = -E_\theta \left(\frac{\partial^2}{\partial\theta^2} \log f(X, \theta) \right).$$

A 6. 各 $\theta \in \Theta$ に対して, コンパクトな θ_0 の近傍 Θ_0 および関数 $G(x)$ が存在して, すべての $\theta \in \Theta_0$ に対して

$$\left| \frac{\partial^i}{\partial\theta^i} \log f(X, \theta) \right| \leqq G(X), \quad i = 1, 2, 3, 4, 5.$$

$$\sup_{\theta \in \Theta_0} E_\theta \left(G(X) \right)^4 < \infty.$$

A 7.　すべての $j\ (\leqq 5)$, $j = i_1 + i_2 + \cdots + i_5$, $i_j = 0, 1, 2, \cdots, 5$,

に対して,

$$E_\theta \left\{ \frac{\partial^{i_1}}{\partial\theta^{i_1}} \log f(x, \theta) \cdot \frac{\partial^{i_2}}{\partial\theta^{i_2}} \log f(x, \theta) \cdot \cdots \cdot \frac{\partial^{i_5}}{\partial\theta^{i_5}} \log f(x, \theta) \right\}$$

は $\theta \in \Theta$ に関して, 4回連続微分可能で, その微分係数は θ に関して連続である.

A 8.　θ の最尤推定量 $\hat{\theta}$ が一意に定まる.

すべての $\theta_0 \in \Theta$ に対して θ_0 の近傍 Θ_0 が存在して, すべての $\theta \in \Theta_0$ で一様に $o(n^{-1})$ まで Edgeworth 展開可能である θ の推定量 T の全体を $\mathcal{T}$ とするとき以下の定理が成り立つ.

定理　推定量 $T \in \mathcal{T}$ に対して, 関数 $c_0(\theta)$, $c_1(\theta)$, $t(\theta)$, $c_2(\theta, t(\theta))$ が存在して, 統計量

$$\begin{aligned}\bar{\theta} = \hat{\theta} &+ \frac{1}{n^{1/2}} c_0(\hat{\theta}) \\ &+ \frac{1}{n} \left\{ \left(c_0(\hat{\theta}) - \frac{1}{2} t(\hat{\theta}) \right) I_{1,1}^{-1}(\hat{\theta}) Z_2(\hat{\theta}) + c_1(\hat{\theta}) \right\} \\ &+ \frac{1}{n^{3/2}} \left\{ \left(c_0(\hat{\theta}) - \frac{1}{2} t(\hat{\theta}) \right) I_{1,1}^{-2}(\hat{\theta}) \left(K_{2,2}(\hat{\theta}) - I_{1,1}^{2}(\hat{\theta}) \right) \right. \\ &\qquad \left. + c_2(\hat{\theta}, t(\hat{\theta})) \right\}\end{aligned}$$

は次式を満たす:

すべての $x_1, x_2 \geqq 0$ に対して,

$$P_{\theta_0}(-x_1 \leqq n^{1/2}(T-\theta) \leqq x_2)$$
$$\leqq P_{\theta_0}(-x_1 \leqq n^{1/2}(\bar{\theta}-\theta) \leqq x_2) + o(n^{-1})$$

ただし

$$Z_2(\theta) = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial \theta^2} \log f(x, \theta) + I_{1,1}(\theta) \right)$$

$$K_{2.2}(\theta) = E_\theta \left(\frac{\partial^2}{\partial \theta^2} \log f(x, \theta) \cdot \frac{\partial^2}{\partial \theta^2} \log f(x, \theta) \right),$$

定理の証明

$$W_n(\theta, t) = \sum_{i=1}^{n} \log f(x_i, \theta + \frac{t}{n^{1/2}}) - \sum \log f(x_i, \theta) - b(\theta, t, \frac{1}{n^{1/2}})$$

とし，$W_n(\theta, t) = 0$ の解を $\theta^*(t)$ とすれば $C_0(\theta, t)$, $C_1(\theta, t)$, $C_2(\theta, t)$ が存在して，確率 $1-o_p(n^{-1})$ で，

$$\theta^*(t) = \theta + \frac{1}{n^{1/2}}(I_{1.1}^{-1}(\theta)Z_1(\theta) + C_0(\theta, t) - t)$$
$$+ \frac{1}{n}\{I_{1.1}^{-2}(\theta)Z_1(\theta)Z_2(\theta) + \frac{1}{2}I_{1.1}^{-3}(\theta)J_3(\theta)Z_1^2(\theta)$$
$$+(C_0(\theta, t) - \frac{t}{2})I_{1.1}^{-1}(\theta)(Z_2(\theta) - I_{1.1}^{-1}(\theta)J_{2.1}(\theta)Z_1(\theta))$$
$$+C_0'(\theta, t)I_{1.1}^{-1}(\theta)Z_1(\theta) + C_1(\theta, t)\}$$
$$+ \frac{1}{n^{3/2}}[I_{1.1}^{-3}(\theta)Z_1(\theta)Z_2^2(\theta) + \frac{3}{2}I_{1.1}^{-4}(\theta)J_3(\theta)Z_1^2(\theta)Z_2(\theta)$$
$$+\frac{1}{2}I_{1.1}^{-3}(\theta)Z_1^2(\theta)Z_3(\theta) + \{\frac{1}{2}I_{1.1}^{-5}(\theta)J_3^2(\theta) + \frac{1}{6}I_{1.1}^{-4}(\theta)K_4(\theta)\}Z_1^3(\theta)$$
$$+ \{-(C_0(\theta, t) - \frac{t}{2})I_{1.1}^{-3}(\theta)(5J_{2.1}(\theta) + J_{1.1.1}(\theta))$$
$$+2C_0'(\theta, t)I_{1.1}^{-2}(\theta)\}Z_1(\theta)Z_2(\theta) + (C_0(\theta, t) - \frac{t}{2})I_{1.1}^{-2}(\theta)Z_1(\theta)Z_3(\theta)$$
$$+ \{\frac{1}{2}(C_0(\theta, t) - \frac{t}{2})I_{1.1}^{-4}(\theta)J_{2.1}(\theta)(-J_3(\theta) + 6J_{2.1}(\theta) + 2J_{1.1.1}(\theta))$$

$$-\frac{1}{2}(C_0(\theta,t)-\frac{t}{2})I_{1.1}^{-3}(\theta)(2K_{3.1}(\theta)+K_{2.2}(\theta)+K_{2.1.1}(\theta))$$
$$+\frac{1}{2}C_0'(\theta,t)I_{1.1}^{-3}(\theta)(J_3(\theta)-2J_{2.1}(\theta))$$
$$+\frac{1}{2}C_0''(\theta,t)I_{1.1}^{-2}(\theta)\ \}\ Z_1^2(\theta)+(C_0(\theta,t)-\frac{t}{2})I_{1.1}^{-2}(\theta)Z_2^2(\theta)$$
$$+\ \{(C_0(\theta,t)-\frac{t}{2})C_0'(\theta,t)-(C_0(\theta,t)-\frac{t}{2})^2 I_{1.1}^{-1}(\theta)J_{2.1}(\theta)+C_1(\theta,t)\ \}$$
$$\times I_{1.1}^{-1}(\theta)(Z_2(\theta)-I_{1.1}^{-1}\theta)J_{2.1}(\theta)Z_1(\theta))+C_1'(\theta,t)I_{1.1}^{-1}(\theta)Z_1(\theta)$$
$$+\frac{1}{6}I_{1.1}^{-1}(\theta)\ \{3t(C_0(\theta,t)-t)+3(C_0(\theta,t)-t)^2+t^2\}$$
$$\times(Z_3(\theta)-I_{1.1}^{-1}(\theta)K_{3.1}(\theta)Z_1(\theta))\ +C_2(\theta,t)\ ].$$

m.l.e. の言葉で言い替えれば,

$$\theta^*(t)=\hat\theta+\frac{1}{n^{1/2}}(C_0(\hat\theta,t)-t)$$
$$+\frac{1}{n}\{(C_0(\hat\theta,t)-\frac{t}{2})I_{1.1}^{-1}(\hat\theta)Z_2(\hat\theta)+C_1(\hat\theta,t)\}$$
$$+\frac{1}{n^{3/2}}\Big[(C_0(\hat\theta,t)-\frac{t}{2})I_{1.1}^{-2}(\hat\theta)Z_2^2(\hat\theta)$$
$$+I_{1.1}^{-1}(\hat\theta)\{-(C_0(\hat\theta,t)-\frac{t}{2})^2 I_{1.1}^{-1}(\hat\theta)J_{2.1}(\hat\theta)$$
$$+(C_0(\hat\theta\ t)-\frac{t}{2})C_0'(\hat\theta,t)$$
$$+C_1(\hat\theta,t)\ \}\ Z_2(\hat\theta)$$
$$+\frac{1}{6}\{3t(C_0(\hat\theta,t)-t)+3(C_0(\hat\theta,t)-t)^2+t^2\}$$
$$\times I_{1.1}^{-1}(\hat\theta)Z_3(\hat\theta)+C_2(\hat\theta,t)\Big]+o_p(n^{-1}).$$

ところで

$$\theta^{**}(t)=\hat\theta+\frac{1}{n^{1/2}}(C_0(\hat\theta,t)-t)$$
$$+\frac{1}{n}\{(C_0(\hat\theta,t)-\frac{t}{2})I_{1.1}^{-1}(\hat\theta)Z_2(\hat\theta)+C_1(\hat\theta,t)\}$$
$$+\frac{1}{n^{3/2}}\{(C_0(\hat\theta,t)-\frac{t}{2})I_{1.1}^{-2}(\hat\theta)(K_{2.2}(\hat\theta)-I_{1.1}^{2}(\hat\theta))$$
$$+C_2(\hat\theta,t)\}$$

なる統計量を導入すれば,

$$\theta^*(t) = \theta^{**}(t) + o_p(n^{-1}) \quad (\text{in law})$$

であることが分かる。

今推定量Tが，すべての $\theta \in \Theta$ に対して，

$$P_\theta(T \leqq \theta) = g(\theta, \frac{1}{n^{1/2}}) = 1 - P_\theta(T \geqq \theta)$$

を満たすとしよう．このとき，$\theta^{**}(t)$ が，任意の $\theta \in \Theta$ に対し，

$$P_{\theta + t/n^{1/2}}(\theta^{**}(t) \leqq \theta) \leqq g(\theta + \frac{1}{n^{1/2}}, \frac{1}{n^{1/2}}) + o(n^{-1})$$

を満たすならば，不等式

$$P_\theta(T \leqq \theta + \frac{t}{n^{1/2}}) \leqq P_\theta(\theta^{**}(t) \leqq \theta) + o(n^{-1}) \quad (t>0)$$

$$P_\theta(T \geqq \theta + \frac{t}{n^{1/2}}) + o(n^{-1}) \geqq P_\theta(\theta^{**}(t) \geqq \theta) \quad (t<0)$$

が成り立つ．

従って $\hat{\theta}(t) = \theta^{**}(t) + \frac{t}{n^{1/2}}$ と置けば，任意の $\theta \in \Theta$ に対して，

$$P_\theta(T \leqq \theta + \frac{t}{n^{1/2}}) \leqq P_\theta(\hat{\theta}(t) \leqq \theta + \frac{t}{n^{1/2}}) + o(n^{-1}) \quad (t>0)$$

$$P_\theta(T \geqq \theta + \frac{t}{n^{1/2}}) + o(n^{-1}) \geqq P_\theta(\hat{\theta}(t) \geqq \theta + \frac{t}{n^{1/2}}) \quad (t<0)$$

となる。

$T \in \mathcal{F}$ に対して，$\theta \in \Theta_0$ における $n^{\frac{1}{2}}(T - \theta)$ の i-th cumulant

$K_i(\theta)$ を

$$K_1(\theta, \frac{1}{n^{1/2}}) = \frac{1}{n^{3/2}} K_{11}(\theta) + \frac{1}{n} K_{12}(\theta) + o(n^{-1}),$$

$$K_2(\theta, \frac{1}{n^{1/2}}) = I_{1.1}^{-1}(\theta) + \frac{1}{n^{1/2}} K_{21}(\theta) + \frac{1}{n} K_{22}(\theta) + o(n^{-1}),$$

$$K_3(\theta, \frac{1}{n^{1/2}}) = \frac{1}{n^{1/2}} K_{31}(\theta) + \frac{1}{n} K_{32}(\theta) + o(n^{-1}),$$

$$K_4(\theta, \frac{1}{n^{1/2}}) = \frac{1}{n} K_{42}(\theta) + o(n^{-1}),$$

$$K_i(\theta) = o(n^{-1}) \quad (i \geqq 5)$$

で表わそう. $T\in\mathfrak{F}$であるから, Tに対応して定まる$\hat{\theta}(t)$において$C_0(\theta,t)$, $C_1(\theta,t)$はtと無関係であるから, 以後$C_0(\theta,t)$, $C_1(\theta,t)$に対して, $C_0(\theta)$, $C_1(\theta)$を用いる.

ところで推定量$T\in\mathfrak{F}$が second order admissible であるならば

$$K_{10}(\theta)=C_0(\theta),\qquad K_{20}(\theta)=I_{1\cdot1}^{-1}(\theta),$$

$$K_{11}(\theta)=C_1(\theta)-\frac{1}{2}I_{1\cdot1}^{-2}(\theta)(J_{2\cdot1}(\theta)+J_{1\cdot1\cdot1}(\theta)),$$

$$K_{21}(\theta)=2I_{1\cdot1}^{-1}(\theta)K_{10}'(\theta),$$

$$K_{31}(\theta)=-I_{1\cdot1}^{-3}(\theta)(3J_{2\cdot1}(\theta)+2J_{1\cdot1\cdot1}(\theta)),$$

ただし $K_{10}'(\theta)=\frac{d}{d\theta}K_{10}(\theta)$ かつ すべての$t\neq 0$に対して,

$$\begin{aligned}
&I_{1\cdot1}^{5/2}(\theta)\{-A_{31}(\theta)+(K_{42}^{*}(\theta)-K_{42}(\theta))I_{1\cdot1}(\theta)\}t^2\\
&+I_{1\cdot1}^{5/2}(\theta)\ [4(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))\\
&\quad+\{4A_{31}(\theta)-3(K_{42}^{*}(\theta)-K_{42}(\theta))I_{1\cdot1}(\theta)\}C_0(\theta)]\,t\\
&+I_{1\cdot1}^{3/2}(\theta)\ [12(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta))-3(K_{42}^{*}(\theta)-K_{42}(\theta))I_{1\cdot1}(\theta)\\
&\qquad-8(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))I_{1\cdot1}(\theta)C_0(\theta)\\
&\qquad+\{-4A_{31}(\theta)+3(K_{42}^{*}(\theta)-K_{42}(\theta))I_{1\cdot1}(\theta)\}I_{1\cdot1}(\theta)C_0^2(\theta)]\leqq 0
\end{aligned}$$

ただし

$$\begin{aligned}
A_{20}(C_0'(\theta),C_1'(\theta),\theta)=&-I_{1\cdot1}^{-3}(\theta)(K_{3\cdot1}(\theta)+4K_{2\cdot1\cdot1}(\theta)+K_{1\cdot1\cdot1\cdot1}(\theta)+I_{1\cdot1}^{2}(\theta))\\
&+\frac{1}{2}I_{1\cdot1}^{-4}(\theta)(7J_{2\cdot1}^{2}(\theta)+14J_{2\cdot1}(\theta)J_{1\cdot1\cdot1}(\theta)+5J_{1\cdot1\cdot1}^{2}(\theta))\\
&+(C_0'(\theta))^2I_{1\cdot1}^{-1}(\theta)+2C_1'(\theta)I_{1\cdot1}^{-1}(\theta),
\end{aligned}$$

$$\begin{aligned}
A_{30}(C_0'(\theta),C_0''(\theta),\theta)=&-3C_0'(\theta)I_{1\cdot1}^{-3}(\theta)(3J_{2\cdot1}(\theta)+2J_{1\cdot1\cdot1}(\theta))\\
&+3C_0''(\theta)I_{1\cdot1}^{-2}(\theta),
\end{aligned}$$

$$A_{31}(\theta)=3I_{1\cdot1}^{-4}(\theta)\ \{I_{1\cdot1}(\theta)(K_{2\cdot2}(\theta)-I_{1\cdot1}^{2}(\theta))-J_{2\cdot1}^{2}(\theta)\}.$$

$$\begin{aligned}
K_{42}^{*}(\theta)=&-I_{1\cdot1}^{-4}(\theta)(4K_{3\cdot1}(\theta)+12K_{2\cdot1\cdot1}(\theta)+3K_{1\cdot1\cdot1\cdot1}(\theta)+3I_{1\cdot1}^{2}(\theta))\\
&+12I_{1\cdot1}^{-5}(\theta)(2J_{2\cdot1}(\theta)+J_{1\cdot1\cdot1}(\theta))(J_{2\cdot1}(\theta)+J_{1\cdot1\cdot1}(\theta))\}
\end{aligned}$$

上の不等式から

$$K^{*}_{42}(\theta)-K_{42}(\theta)\leqq 0.$$

$$\frac{1}{4}I_{1.1}(\theta)\{A_{31}(\theta)-(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}A^{2}(\theta)$$

$$+12(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta))-3(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)$$

$$-8(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))I_{1.1}(\theta)C_0(\theta)$$

$$+\{-4A_{31}(\theta)+3(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}I_{1.1}(\theta)C_0^{2}(\theta)\leqq 0,$$

ただし

$$A(\theta)=\frac{4(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))+(4A_{31}(\theta)-3(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta))C_0(\theta)}{-A_{31}(\theta)+(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)}$$

$t(\theta)=-\frac{1}{2}A(\theta)$ とおき, $\tilde{\theta}(t(\hat{\theta}))$ とする。 このとき,

$$\tilde{\theta}(t(\hat{\theta}))=\tilde{\theta}(t(\theta))+o_p(n^{-1})\quad(\textit{in law})$$

であって,

$$P_{\theta}(n^{1/2}(T-\theta)\leqq x)-P_{\theta}(n^{1/2}(\tilde{\theta}(t(\hat{\theta}))-\theta)\leqq x)$$

$$=\frac{1}{24n}\phi(I_{1.1}^{1/2}(\theta)(x-C_0(\theta)))I_{1.1}^{5/2}(\theta)x$$

$$\times[(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)(x+\frac{1}{2}A(\theta))^{2}$$

$$-\frac{1}{4}\{-A_{31}(\theta)+(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}A^{2}(\theta)$$

$$+12(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta))I_{1.1}^{-1}(\theta)-3(K^{*}_{42}(\theta)-K_{42}(\theta))$$

$$-8(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))C_0(\theta)$$

$$-\{4A_{31}(\theta)-3(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}C_0^{2}(\theta)]+o(n^{-1})$$

従って,

$$P_{\theta}(n^{1/2}(T-\theta)\leqq x)\leqq P_{\theta}(n^{1/2}\{\tilde{\theta}(t(\hat{\theta}))-\theta\}\leqq x)+o(n^{-1})\qquad(x\geqq 0)$$

$$P_{\theta}(n^{1/2}(T-\theta)<x)+o(n^{-1})\geqq P_{\theta}(n^{1/2}\{\tilde{\theta}(t(\hat{\theta}))-\theta\}<x)\qquad(x\leqq 0)$$

ゆえに すべての $x_1, x_2 \geqq 0$ に対して,

$$P_{\theta}(-x_1\leqq n^{1/2}(T-\theta)\leqq x_2)\leqq P_{\theta}(-x_1\leqq n^{1/2}(\tilde{\theta}(t(\hat{\theta}))-\theta)\leqq x_2)+o(n^{-1})$$

が成り立つ.