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Singular set of stable mappings from 4-manifolds into 3-manifolds

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1. Introduction

The present note contains the investigation of the topology of a singular set of stable mappings. It is a continuation of the author's note [3], where analogous theorems on the topology of singular sets of certain mapping are formulated. Our main concern is the stable mapping from 4-manifolds into 3-dim euclidean space, $f: M \to \mathbb{R}^3$. Since f is stable, possible singularity types are the followings ,

1) $(u,x,y,z) \longrightarrow (u,x,y^2 \pm z^2)$, fold point ($A_1$-type)
2) $(u,x,y,z) \longrightarrow (u,x,y^3+uy\pm z^2)$, cusp point ($A_2$-type)
3) $(u,x,y,z) \longrightarrow (u,x,y^4+uy^2+xy\pm z^2)$, swallow tail($A_3$-type)

In this case it is easy to see that the singular set, S(f), of the mapping f is 2-dim submanifolds of M^4. Then it arises a question to what extent the structure of S(f) is determined by the mapping and topology of M^4. We are concerned with the location of S(f) in M^4.

2. Congruence formula

To extract embedding phenomina of S(f) in M^4, we will study the structure of the normal bundle of S(f) in M. To be precise, we will formulate the relation between self-intersection number of S(f) and signature of M. We set several assumptions for simplicity from now on: M is oriented and first integral homology group of M vanishes. This means that torsion of second (co)homology is free, for $H^2(M;Z) = \mathrm{Hom}(H_2(M;Z),Z) \oplus \mathrm{Ext}(H_1(M;Z),Z)$, which follows from universal coefficient theorem.

For a stable mapping $f:M^4 \to \mathbb{R}^3$, we can prove the following formula,

$$\sigma(M) \equiv -S(f)\cdot S(f) \mod 4 \qquad (1)$$

Here $\sigma(M)$ is defined as the signature of the cup product bilinear form on the 2-nd cohomology group and the dot stands for the self intersection number of S(f) in M^4. Before proving formula (1), we need a key result. For our mapping f it holds,

$$\chi(M) \equiv \chi(S(f)) \mod 2 \qquad (2)$$

where χ denotes the Euler characteristic. The requisite technique of the proof is to consider the orthogonal projection from $\mathbb{R}^3$ to an one dimensional linear subspace L of $\mathbb{R}^3$ (See [1]). We can regard L as a line through an origin of R^3 i.e. an element of $\mathbb{RP}^2$, 2-dim projective space. Let $A_k(f)$ stand for an A_k-type singular set of f ($1 \leq k \leq 3$). It follows from the genericity of the above projection π: $\mathbb{R}^3 \to L$ that for almost every line of $\mathbb{RP}^2$ the composed mapping $\pi \circ f$ and $\pi \circ f|A_k(f)$ ($1 \leq k \leq 3$) are Morse functions. Let #C(g) be the number of critical points of a Morse function g: $M \to \mathbb{R}$. Since the closure of $A_2(f)$, $\overline{A_2}(f)$, is a disjoint union of a circle, $\#C(\pi \circ f|\overline{A_2}(f))$ is even. Moreover, it is easy to see

$$\#C(\pi \circ f) \equiv \#C(\pi \circ f|A_1(f)) \mod 2,$$
$$\#C(\pi \circ f|S(f)) = \#C(\pi \circ f|A_1(f)) + \#C(\pi \circ f|\overline{A_2}(f))$$
$$\equiv \#C(\pi \circ f|A_1(f)) \mod 2.$$

From the Morse theory the Euler characteristic of a compact manifold is mod2 congruent with the number of critical points of the Morse function over the manifold. Thus formula (2) follows. Next, the following is well known as the generalized Whitney congruence (See [2]). Let F^2 be a characteristic surface of M^4. If the first integral homology group of M vanishes, it holds

$$\sigma(M) \equiv F \cdot F + 2\chi(F) \mod 4. \qquad (3)$$

F is characteristic if the mod 2 cycle $[F]_2$ is dual to the 2-nd Stiefel-Whitney class $w_2(M)$. In our situations the following fact is well known (See [4]). S(f) is a mod 2 cycle and its Poincarè dual coincides with $w_2(M)$. Thus formula (1) automatically follows from (2) and (3). Formula (1) merely suggests certain kind of restriction between the topology of a domain and singular set and does not completely elucidate the location of S(f).

3. Orientability of singular sets

We should need to consider the orientability of S(f). For our mapping f we have an immediate result concerning the orientability of S(f).

(i) If the signature of M^4 is odd, then S(f) must be unorientable. More precisely, S(f) contains unorientable surfaces with odd genus , which follows from classification of closed surfaces and formula (2).

The case where $\sigma(M)$ is even is more complicated and subtile. We will continue to study the orientability of S(f), assuming $\sigma(M)$ is even. Let φ be a smooth embedding of S(f) into M , φ: S(f) $\rightarrow$ M. If φ^*: H (M;Z/2) $\rightarrow$ H (S(f);Z/2) is the homomorphism, induced by φ, we assume φ^* is an injection for a while, then we obtain the following bundle isomorphism

$$\tau(S(f)) \oplus \nu_\varphi \cong \varphi^* \tau(M),$$

where ν_φ denotes the normal bundle of S(f) in M. Then it holds,

$$\begin{aligned} w(\varphi^*\tau(M)) &= (1+w_1(S(f))+w_2(S(f)))(1+w_1(\nu_\varphi)+w_2(\nu_\varphi)) \\ &= 1+(w_1(S(f))+w_1(\nu_\varphi))+(w_2(S(f))+w_2(\nu_\varphi)+w_1(S(f))w_1(\nu_\varphi)) \\ &\quad +(w_2(S(f))w_1(\nu_\varphi)+w_2(\nu_\varphi)w_1(S(f)))+w_2(S(f))w_2(\nu_\varphi). \qquad (4) \end{aligned}$$

If S(f) is orientable, then $w_1(S(f))$ and $w_2(S(f))$ vanishes. Thus we have,

$$\begin{aligned} \varphi^* w_2(M^4) &= w_2(\varphi^*\tau(M)) \\ &= w_2(S(f))+w_2(\nu_\varphi)+w_1(S(f))w_1(\nu_\varphi) \\ &= w_2(\nu_\varphi). \qquad (5) \end{aligned}$$

On the other hand, from formula (1) the self-intersection number of S(f) is even since $\sigma(M)$ is even. Hence,

$$\langle w_2(\nu_\varphi), [S(f)]_2 \rangle \equiv S(f)\cdot S(f) \equiv 0 \mod 2.$$

If the mod 2 cycle $[S(f)]_2$ is a zero element, then $w_2(M)$ vanishes from the fact which we state above. Otherwise, $w_2(\nu_\varphi)$ vanishes. Also in this case by equation (5) $w_2(M)$ vanishes from the injectivity of φ^*. Thus we have proven that M is spin. From the classical Rochlin's theorem , the signature of M must be divisible by 16. Therefore, assume $\sigma(M)$ is even and not divisible by 16, if φ^* is an injection then S(f) is unorientable.

4. Branched covering space

In this section we will study the general case in which the target space is an arbitrary oriented 3-manifold. For any stable mapping $f: M^4 \to N^3$, the mod2 cycle $[S(f)]_2$ is also dual to the 2-nd Stiefel-Whitney class of M since any orientable 3-manifold is parallelizable and vanish the contributions of characteristic classes of $f^* T(N)$ in the Thom polynomial. Accordingly, it is expected that if M is a spin manifold, namely, $w_2(M)=0$, the embedding phenomina of S(f) should be more accurately evaluated. We will explain our trial to calculate the self-intersection number of S(f) in M hereafter. Since M is spin, $[S(f)]_2$ is a zero element of $H_2(M;Z/2)$. Suppose S(f) is orientable. Non-orientable case will be treated afterwards. Let $\varphi: S(f) \to M$ be a smooth embedding with $\varphi^*[S(f)] = u \in H^2(M;Z)$. Then we can define the divisibility of S(f),

$$div(S(f)) = \max\{n \in Z;\ u=nv \text{ for some } v \in H_2(M;Z)\}.$$

We have an immediate result from this definition, considering the following homomorphism

$$H_2(M;Z) \longrightarrow H_2(M;Z/2) \xrightarrow{\cong} H^2(M;Z/2),$$

where the first arrow is mod2Z coefficient homomorphism and second one is Poincaré duality isomorphism. We claim div(S(f)) is even, and we set div(S(f))=2m. This follows from the fact that $[S(f)]_2$ is a zero element since M is spin. Then the integral homology class [S(f)] is dual to 2mx for some x of $H^2(M;Z)$ by Poincaré duality. Let E be the complex line bundle over M such that $c_1(E)=x$, where $c_1(E)$ denotes the first Chern class of E. Then $c_1(E^{2m})=c_1(E\otimes \cdots \otimes E)= 2mx$, where E^{2m} is the 2m-fold tensor product bundle of E. Choose a cross-section $s: M \to E^{2m}$ of the bundle E^{2m} which is transeversal to the zero section M and equal to zero exactly on S(f) embedded in M. Such a cross section always exists by Thom's transeversality theorem. We define $p_{2m}: E \to E^{2m}$ by $p_{2m}(v) = v\otimes \cdots \otimes v$. p_{2m} is a branched covering of order 2m branched along the zero section,M. Take $\widetilde{M} = p_{2m}^{-1}\, s(M)$. The covering map $\pi: \widetilde{M} \to M$ is the composition of $p_{2m}|M$ with the bundle projection $E^{2m} \to M$. $\widetilde{M}$ is a 2m-fold branched covering branched along the zeroes S(f) of s. It is a diffeomorphism on S(f) and an usual covering on the complement of S(f) in $\widetilde{M}$ and M.

5. Algebraic invariant and its application

Let H be a finite dimensional real vector space and t a linear transformation of H such that $t^n=\mathrm{id}$. Let f be a quadratic form over H, invariant under t. Define the polynomials by the formula,

$$P_k(z)=\begin{cases} z-1 & \text{if } k=0 \\ (z-\varsigma^k)(z-\varsigma^{-k}) & \text{if } 1\leq k<n/2 \\ z+1 & \text{if } k=n/2 \end{cases} \tag{6}$$

, where $\varsigma=\exp(2\pi i/n)$.

Then it obviously holds,

$$z^n-1=\prod_{k=0}^{[n/2]} P_k(z). \tag{7}$$

This expansion corresponds to a decomposition of H in the direct sum of subspaces $\mathrm{Ker}P_k(z)$ $(0\leq k\leq[n/2])$. Then we define t-signature of a quadratic form f,

$$\sigma(f,t)=\sum_{k=0}^{[n/2]} a(k)\cos 2k\pi/n, \tag{8}$$

where a(k) denotes the signature of f over $\mathrm{Ker}P_k(t)$. If t=id, then $\sigma(f,t)$ denotes the ordinary signature $\sigma(f)$ of a quadratic form f. Moreover, we define the numbers $\hat{a}(0),\ldots,\hat{a}(n-1)$ by

$$\hat{a}(k)=\begin{cases} a(k) & \text{if } k=0,n/2 \\ a(k)/2 & \text{if } 1\leq k<n/2 \\ a(n-k)/2 & \text{if } n/2<k\leq n-1 \end{cases} \tag{9}$$

It is easy to see, $\sigma(f,t)=\sum_{k=0}^{n-1}\hat{a}(k)\varsigma^k$. (10)

Subspaces $\mathrm{Ker}P_k(t)$ $(0\leq k\leq[n/2])$ are pairwise orthogonal with respect to a quadratic form f. Thus we have

$$\sigma(f,t^s)=\sum_{k=0}^{n-1}\hat{a}(k)\varsigma^{ks}=\sum_{k=0}^{[n/2]}a(k)\cos 2ks\pi/n \quad (s=0,\ldots,n-1) \tag{11}$$

and in particular, since $\sigma(f)=\sigma(f,\mathrm{id})$

$$\sigma(f)=\sum_{k=0}^{n-1}\hat{a}(k)=\sum_{k=0}^{[n/2]}a(k). \tag{12}$$

Since

$$\sum_{k=0}^{n-1}\varsigma^{ks}\cdot\varsigma^{-ks_1}=\begin{cases} n & \text{if } s=s_1 \\ 0 & \text{if } s\neq s_1 \end{cases}$$

we have

$$\hat{a}(k)=1/n\sum_{s=0}^{n-1}\sigma(f,t^s)\varsigma^{-ks}\quad(k=0,\ldots,n-1)$$

$$=1/n\{\sigma(f)+\sum_{s=1}^{n-1}\sigma(f,t^s)\varsigma^{-ks}\}\tag{13}$$

and

$$a(0)=1/n\sum_{s=0}^{n-1}\sigma(f,t^s)=1/n\{\sigma(f)+\sum_{s=1}^{n-1}\sigma(f,t^s)\}\tag{14}$$

Combining (13) and (14),

$$\hat{a}(k)=a(0)-1/n\sum_{s=1}^{n-1}\sigma(f,t^s)(1-\varsigma^{-ks})$$

$$=a(0)-2/n\sum_{s=1}^{n-1}\sigma(f,t^s)\sin^2 ks\pi/n\ .\tag{15}$$

In the previous section we have constructed a 2m-fold branched covering $\P:\tilde{M}\to M$ branched along $S(f)$. Consider the homomorphism $\P^*$: $H^2(M;\mathbb{R})\to H^2(\tilde{M};\mathbb{R})$, induced by $\P$. $\P^*$ isomorphically maps $H^2(M;\mathbb{R})$ on the set of $H^2(\tilde{M};\mathbb{R})^{\mathbb{Z}_{2m}}$, which are fixed with respect to $\mathbb{Z}_{2m}$ action. Consequently, $H^2(M;R)\cong \operatorname{Ker}P_0(t)$. Thus we have $a(0)=\sigma(M)$. In our situation there is an adaption of Atiyah-Singer G-signature theorem ,[5].

$$\sigma(f,t^s)=e[F]/n\sin^2\frac{s}{n}\pi\quad(s=1,\ldots,n-1),\tag{16}$$

where F is the fixed point set of the diffeomorphism $h:M\to M$ of period n and $e[F]$ denotes the self-intersection number of F in M. Accordingly, we have, applying (16) to (15) in our statement

$$\hat{a}(k)=\sigma(M)-2/n^2 S(f)\cdot S(f)\sum_{s=1}^{n-1}\sin^2\frac{ks}{n}\pi/\sin^2\frac{s}{n}\pi\tag{17}$$

We can easily show, considering second difference equation

$$\sum_{s=1}^{n-1}\sin^2\frac{ks}{n}\pi/\sin^2\frac{s}{n}\pi=k(n-k)\tag{18}$$

Thus we have

$$\hat{a}(k)=\sigma(M)-2k(n-k)S(f)\cdot S(f)/n^2.\tag{19}$$

In our case $H=H^2(\tilde{M};R)$ and $\sigma(f)=\sigma(\tilde{M})$. Hence we obtain from (12) and (19),

$$\sigma(\tilde{M})=\sum_{k=0}^{n-1}\hat{a}(k)$$
$$=n\,\sigma(M)-2S(f)\cdot S(f)/n^2\sum_{k=0}^{n-1}k(n-k)$$

$$= n\,\sigma(M) - (n^2-1)S(f)\cdot S(f)/3n \qquad (20)$$

This is Hirzebruch's formula, [6].
Since n=2m,

$$6m\,\sigma(\tilde{M}) = 12m^2\sigma(M)-(4m^2-1)S(f)\cdot S(f). \qquad (21)$$

M is spin, the signature of M is divisible by 16. If we can prove that $\tilde{M}$ is also spin,i.e. $\pi^*(w_2(M))= w_2(\tilde{M})$, we can calculate the self-intersection number of S(f). Since div(S(f)) is even, we have

$S(f)\cdot S(f) \equiv 0 \mod 4$ if m=1,

$S(f)\cdot S(f) \equiv 0 \mod 16$ if m=2,

$S(f)\cdot S(f) \equiv 0 \mod 36$ if m=3 and so on.

Applying equation (21) to this result,

$S(f)\cdot S(f) \equiv 0 \mod 32$

$S(f)\cdot S(f) \equiv 0 \mod 64$

$S(f)\cdot S(f) \equiv 0 \mod 288$ and so on.

If S(f) is non-orientable, we can construct a double branched covering branched along S(f). Thus this case is contained in m=1.

References

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