数理解析研究所講究録
第 726 巻 1990 年 139-152

Remarks on Specified Strong ε-Cores of Games

特性関数型ゲームの強コアに関する注意

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1. Introduction and Preliminaries.

This is a note on the game space and its dual space, which were originally examined by Bondareva [1] and Shapley [7], and later studied extensively by Maschler/Peleg/Shapley [3]. In Rosenmuller [4] and other papers, solutions of extreme games are examined. It will be considered later to rewrite the characterization of extreme games given in [4] in terms of dual variables

The motivation for this note is a fact that a subconvex game has a nonempty core, which was shown in Sharkey [11] by saying that the core of a subconvex game is large. Alternatively we can see this via the inclusion relation between dual sets. Furthermore, we can see relations of extreme points of these dual sets, from which we can interpret the depth of a minimal balanced set. By considering dual sets more we define a generalization of balanced sets. Then we show that the characteristic-function of a game satisfies a system of balanced inequalities if and only if some strong epsilon-core is nonempty. This is a generalization of Theorem 1 of [7]. The epsilon defined in this note is a piecewise-linear function of characteristic-function. This epsilon-core may measure the strength of a property that a game has.

We begin with giving basic definitions and the notation. Let $N = \{1, 2, \ldots, n\}$ be the set of *players* . Any subset of N is called a *coalition* . A pair $\Gamma \equiv (N,v)$ is called a *game* where v is a real-valued function on 2^N with $v(\phi) = 0$. v is said to be the *characteristic function* of Γ. We assume v always takes finite values. For $S \subset N$, a *subgame* , written as $\Gamma|_S = (S,v|_S)$, of Γ is a game such that S is the set of players and $v|_S$ is the restriction of v to 2^S. The zero-normalization of v is written as $\mathcal{N}v$, i.e., $\mathcal{N}v(S) = v(S) - \Sigma_{i\in S} v(\{i\})$ for all $S \subset N$. For any finite set K, R^K is the $\|K\|$-dimensional Euclidean space whose coordinates are indexed by the elements of K, where $||K||$ is the number of elements in K. For a game $\Gamma = (N,v)$, let $X(\Gamma) \equiv \{x \in R^N : x(N) = v(N)\}$, where $x(\bullet)$ is a short notation for $\Sigma_{i \in \bullet} x_i$. For convenience we let $x(\phi) = 0$. Any element of $X(\Gamma)$ is called a *preimputation* for Γ.

The *core* of a game $\Gamma = (N,v)$ is defined by :

$$C(\Gamma) \equiv \{x \in X(\Gamma) : x(S) \geq v(S) \text{ for all } S \subset N\}.$$

Let $S_1, \ldots, S_p$ be distinct, nonempty, proper subsets of N. The set $B = \{S_1, \ldots, S_p\}$ is said to be a *balanced set* on N if there exists positive coefficients $w_1, \ldots, w_p$ such that

(1.1) $$\Sigma_{i \in S_j} w_j = 1 \text{ for all } i \in N.$$

Theorem 1.([1],[7]) A necessary and sufficient condition that a game $\Gamma = (N,v)$ has a nonempty core is that for every balanced set $B = \{S_1, \ldots, S_p\}$ on N it satisfies :

(1.2) $$v(N) \geq \sum_{j=1}^{p} w_j \; v(S_j)$$

where $w_1, \ldots, w_p$ are associated positive coefficients[1]

This theorem is a consequence of the duality theorem in linear programming, and the basis for this note. A purpose of this note is to extend this theorem. Thus we define extensions of the core and balanced sets.

Let $\Gamma = (N,v)$ be a game and ε be a real number. The *strong ε-core* of Γ is defined by :

$$C_\varepsilon(\Gamma) \equiv \{x \in X(\Gamma) : x(S) \geq v(S) - \varepsilon \text{ for all } S \neq N, \phi\}.$$

If $\varepsilon = 0$ then the strong ε-core reduces to the core. For $S \subset N$, $S \neq \phi$, we let $\wp(S) \equiv 2^S \setminus (\{\{i\} : i \in S\} \cup \{S, \phi\})$ and $W(S) \equiv R^{\wp(S)}$. Throughout this note we simply write W(N) as W. For $w \in W$, we let $\underline{w} \equiv \Sigma_{S\in \wp(N)} w_S + \Sigma_{i\in N}[1 - \Sigma_{i\in S,S\in \wp(N)} w_S]$. Any element of W is said to be a *generalized partition* [2] on N. For $w \in W$, if $\Sigma_{i\in S,S\in \wp(N)} w_S \leq 1$ for all $i \in N$, and if $w_S \geq 0$ for all $S \in \wp(N)$, then w is associated with a balanced set B defined by $B = \{S : w_S > 0\} \cup \{\{i\} : 1 - \Sigma_{i\in S,S\in \wp(N)} w_S > 0\}$. Conversely for a balanced set $B = \{S_1, \ldots, S_p\}$ with $w_1, \ldots, w_p$, we define $w_S = w_j$ if $S = S_j$ and $S \in \wp(N)$, and $w_S = 0$ for other $S \in \wp(N)$. Then $\Sigma_{i\in S,S\in \wp(N)} w_S \leq 1$ for all $i \in N$. Thus we let $W^b \equiv \{w \in : w_S \geq 0$, all $S \in \wp(N)$, and $\Sigma_{i\in S,S\in \wp(N)} w_S \leq 1$, all $i \in N\}$. This is the set of all generalized partitions which are associated with balanced sets.

In the next section we investigate properties of subsets of W that characterize classes of games via the duality relation in linear programming. In Section 3 we discuss on an inductive method for constructing balanced sets. In Section 4 we state an extension of Theorem 1. Section 5 consists of remarks.

2. Shapes of Dual Sets

If a class of games with the player set N is defined by a statement, which can be expressed in a form of balanced[3] linear inequalities with respect to characteristic-function, the class is characterized by a subset of W, so that

$$(2.1) \qquad \mathcal{N}v(N) \geq \langle w;v\rangle \equiv \Sigma_{Q\in \wp(N)} w_Q \mathcal{N}v(Q)\ , \text{ all } w \in W^{\alpha},$$

where α is a parameter which indicates some condition and W^{α} is a convex and closed subset of W such that it has a finite number of extreme points and it is invariant under any permutation on N.

For example, we can see in the literature

$\alpha = e$: A game $\Gamma = (N,v)$ is *essential* , i.e., $\mathcal{N}v(N) \geq 0$,
$= b$: A game $\Gamma = (N,v)$ is *balanced* ,
$= c$: A game $\Gamma = (N,v)$ is *convex* [4], i.e.,
$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subset N$.

W^e is a one-point set consisting of the zero vector in W and W^b has already been defined and it is bounded. By Theorem 1 and the definition of W^b, we see that a game has a nonempty core if and only if it is balanced. W^c is not bounded. This is because the convexity, expressed by c, has the following property : If a game $\Gamma = (N,v)$ satisfies a condition expressed by α, then the system of inequalities, (2.1), can be rewritten as the system about a subgame $\Gamma|_S$, for any $S \subset N$, and any subgame $\Gamma|_S$ satisfies it. This property is called the *totality* [5] here for convenience. In general, if a condition expressed by α has the totality then W^α is not bounded. Let W_+ be the nonnegative orthant of W.

Proposition 2. Assume the statement of a condition expressed by α has the totality and includes $Nv(N) \geq 0$. Then $W^\alpha = W^\alpha - W_+$.

In this section, hereafter we examine the shapes of W^b and W^c.

1. <u>The shape of W^c</u> .

When $n > 3$, for $i, j \in N$, $i \neq j$, define $w^{ij} \in W$ by $w^{ij}_{N\setminus\{i\}} = w^{ij}_{N\setminus\{j\}} = 1$, $w^{ij}_{N\setminus\{i,j\}} = -1$, and $w^{ij}_S = 0$ for all other $S \in \wp(N)$. When $n = 3$, for $i, j \in N$, $i \neq j$, define $w^{ij} \in W$ by $w^{ij}_{N\setminus\{i\}} = w^{ij}_{N\setminus\{j\}} = 1$, and $w^{ij}_S = 0$ for all other $S \in \wp(N)$. Let $\kappa \equiv \{(S,T) : S,T \subset N, S \neq T, T\setminus S \neq \phi, S\setminus T \neq \phi, \text{ and } \|S \cup T\| \leq n-1\}$. For $(S,T) \in \kappa$, and for $z \in R^1$, define a vector in W by $d^{S,T}(z) = (d^{S,T}(z,Q))_{Q \in \wp(N)}$, where

$$d^{S,T}(z,Q) = \begin{cases} -z \text{ if } Q = S \text{ or } T, \text{ and } Q \in \wp(N), \\ z \text{ if } Q = S \cup T \text{ or } S \cap T, \text{ and } Q \in \wp(N), \\ 0 \text{ for all other } Q \in \wp(N). \end{cases}$$

Proposition 3. (i)W^c is the convex hull of $\cup_{i \neq j} \cup_\kappa \{w^{ij} - d^{S,T}(z) : z \geq 0\}$.
(ii) $\underline{w} \geq 1$ for all $w \in W^c$.

2. <u>Extreme points of W^b</u> .

W^b is a compact and convex set with a finite number of extreme points. A balanced set on N is called *minimal* if it includes no other balanced set on N. A point in W^b is an extreme point of W^b if and only if it is associated with a minimal balanced set on N^6. We want to express an extreme point as a convex linear combination of points corresponding to generalized partitions with some properties. Before stating it precisely we introduce some notation and definitions. Let **Z** be the set of all the integers. Let $E \equiv \mathbf{Z}^{@}$ ($@ \equiv 2^n$) be the space of all 2^n-tuples of integers $(z_1, \ldots, z_@)$, $z_r \in \mathbf{Z}$, $r = 1, \ldots, @$. An addition in E (denoted by +*) is defined to be :

$$(z_1, \ldots, z_@) +^* (z'_1, \ldots, z'_@) = (z_1+z'_1, \ldots, z_@+z'_@)$$

and a multiplication by scalars in **Z**

$$k(z_1, \ldots, z_@) = (kz_1, \ldots, kz_@), \; k \in \mathbf{Z}.$$

A subtraction (denoted by -*) is naturally defined by combining the addition with the multiplication by -1. E is closed with respect to these operations. For $z^1, z^2 \in E$, $z^1 \geq z^2$ means $z^1_r \geq z^2_r$ for all $r = 1, \ldots, @$, where z^i_r $(i = 1,2)$ is the r-th component. Number all element of 2^N. So $2^N = \{R_1, \ldots, R_@\}$. Identify R_r $(r = 1, \ldots, @)$ with a unit vector in E, i.e., $z \in E$ such that $z_r = 1$ and $z_{r'} = 0$ for $r' \neq r$. Alternatively we can represent any $z \in E$ by $z_1R_1 +^* z_2R_2 +^* \ldots +^* z_@R_@$. Σ^* is the summation operation with respect to +*.

$w \in W$ is called a *signed partition* on N if there exist P and Q such that :

$P = \{P_1, \ldots, P_k\}$ is a partition of N where $P_j \neq \phi$ for $j = 1, \ldots, k$, and $k > 1$.

$Q = \{Q_1, \ldots, Q_k\}$ is a set of subsets which satisfies

$Q_j \subset \bigcup_{\ell=1}^{j-1} P_\ell$ and $P_j \cup Q_j \neq N$ for $j = 1, \ldots, k$, and[7]

(2.2) $\qquad w_S = \|\{j: P_j \cup Q_j = S\}\| - \|\{j: Q_j = S\}\|$ for all $S \in \wp(N)$.

We let $S = (P, Q)$ and $v(S) = \sum_{j=1}^{k}{}^* (P_j \cup Q_j) -^* \sum_{j=1}^{k}{}^* Q_j$. Note that $P_j \cap Q_j = \phi$ for $j = 1, \ldots, k$. By the definition of P and Q we have

(2.3) $\qquad \|\{j: P_j \cup Q_j = \{i\}\}\| - \|\{j: Q_j = \{i\}\}\| + \Sigma_{i \in S, S \in \wp(N)} w_S = 1$ for all $i \in N$.

This is the reason why we call w a signed partition. If $Q_1 = \ldots = Q_k = \phi$ then w is a partition of N.

Let $B = \{S_1, \ldots, S_p\}$ be a balanced set on N associated by $m_1/m, \ldots, m_p/m$, where $m_1, \ldots, m_p$, and m are positive integers.[8] By (1.1), m is determined by $m_1, \ldots, m_p$. Let $M = (m_1, \ldots, m_p)$ and let $\mu(B;M) \equiv m_1S_1 +^* \ldots +^* m_pS_p$.

Theorem 4. Let $m_1, \ldots, m_p$, and m are positive integers. $B = \{S_1, \ldots, S_p\}$ is a balanced set on N associated by $m_1/m, \ldots, m_p/m$ if and only if there exist signed partitions on N, $w^1, \ldots, w^t$, defined by $S^1 = (P^1, Q^1), \ldots, S^t = (P^t, Q^t)$ and there exist $m^1, \ldots, m^t$ such that :

(2.4) $\quad m^1 + \ldots + m^t = m$, and

(2.5) $$\mu(B;M) = \sum_{r=1}^{t}{}^{*}\ m^r v(S^r),$$

where $P^r = \{P^r_1, \ldots, P^r_{k_r}\}$ and $Q^r = \{Q^r_1, \ldots, Q^r_{k_r}\}$ for $r = 1, \ldots, t$.

We call (2.5) a *representation* of B by $S^1, \ldots, S^t$. From Theorem 4, we have $w = (m^1w^1 + \ldots + m^tw^t)/m$ by dividing both sides of (2.5) by m and applying (2.2). The proof of Theorem 4 gives a procedure to get a representation of a balanced set by a family of signed partitions.

We have a question : Let B be a minimal balanced set. Is a representation of B by signed partitions uniquely determined ? Suppose $\{S^1, \ldots, S^t\}$ and $\{S'^1, \ldots, S'^{t'}\}$ gives two representations. We define a relation $F \sim F'$ by :

(2.6)
(i) $t = t'$ and
(ii) There are permutations π and ρ on $\{1, \ldots, t\}$ and N respectively such that $v(S^j) = v(\rho S^{\pi(j)})$ for all $j = 1, \ldots, t$,

where ρS^j is defined by $\rho P_j = \{\rho(i) : i \in P_j\}$ and $\rho Q^j = \{\rho(i) : i \in Q_j\}$. It is easy to see that the uniqueness holds when n = 3. But we have an example of a minimal balanced set such that two representations are not equivalent in the sense of (2.6). Thus a representation of a minimal balanced set is not always determined uniquely.

Suppose B in Theorem 4 is a minimal balanced set. We know associated positive coefficients are uniquely determined and they are rational numbers. Hence it is possible to define the least common denominator of them. Let m be that number. m is called the depth of B. From Theorem 4, the depth is the minimum number of signed partitions which are necessary to represent B.

3. A Discussion on Constructing Balanced Sets

In this section we discuss on an inductive method for constructing balanced sets, applying the arguments in Section 2. We can give an inductive proof of Theorem 4 , using the procedure just mentioned below.

Assume $\mu(B;M) = \sum_{j=1}^{m} {}^{*}\ \nu(S^j)$. Here $B = \{S_1, \ldots, S_p\}$ is a balanced set on N associated by $m_1/m, \ldots, m_p/m$ where $m_1, \ldots, m_p$ and m are positive integers and $M = (m_1, \ldots, m_p)$. $S^1, \ldots, S^m$ are signed partitions such that $S^j = (P^j, Q^j)$, $P^j = \{P^j_1, \ldots, P^j_{kj}\}$, $Q^j = \{Q^j_1, \ldots, Q^j_{kj}\}$ $(j = 1, \ldots, m)$. Compare this representation with (2.4) and (2.5). It may happen that $S^j = S^{j'}$ for some j and j', etc. We construct a balanced set $B(f,x)$ on $N \cup \{n+1\}$ where f and x are defined below. Define $\theta^0 \equiv \{(j,\ell) : Q^j_\ell \neq \phi\}$. Let $\alpha \equiv \{(j,\ell) : 1 \leq \ell \leq k_j, 1 \leq j \leq m\}$. Since $S^1, \ldots, S^m$ gives a representation of B , that is, $\sum_{j=1}^{m} {}^{*}\ \nu(S^j)$ has no negative coefficient, we can define a one-to-one mapping f from θ^0 into α such that

(3.1) $\qquad f(j,\ell) = (j',\ell)$,

where $Q^j_\ell = P^{j'}_{\ell'} \cup Q^{j'}_{\ell'}$. Let $\beta(f) = \alpha \backslash f(\theta^0)$. Note that

(3.2) $\qquad \mu(B;M) = \Sigma^*_{(j,\ell)\in\beta(f)} (P^j_\ell \cup Q^j_\ell)$.

Define $x = (x(j,\ell))_{(j,\ell) \in \beta(f)}$ by $x(j,\ell) = 1$ or 0 for all $(j,\ell) \in \beta(f)$. Let

(3.3) $\quad m(x) \equiv \|\{(j,\ell) : x(j,\ell) = 1\}\|$ and $q(x) \equiv \max(m,m(x))$.

Define $P_+^j(x) = \{P_+{}^j{}_1, \ldots, P_+{}^j{}_{kj+1}\}$ for $1 \le j \le m$ and $P_+^j(x) = \{P_+{}^j{}_1, P_+{}^j{}_2\}$ for $m+1 \le j \le q(x)$ by :

(3.4)
$$P_+{}^j{}_\ell = \begin{cases} \{n+1\} & \text{if } \ell = 1 \text{ and} 1 \le j \le q(x), \\ P^j{}_{\ell-1} & \text{if } \ell \ge 2 \text{ and } 1 \le j \le m, \text{ and} \\ N & \text{if } \ell = 2 \text{ and } m+1 \le j \le q(x). \end{cases}$$

Define $Q_+^j(x) = \{Q_+{}^j{}_1, \ldots, Q_+{}^j{}_{kj+1}\}$ for $1 \le j \le m$ and $Q_+^j(x) = \{Q_+{}^j{}_1, Q_+{}^j{}_2\}$ for $m+1 \le j \le q(x)$ as follows.

(3.5) $\quad Q_+{}^j{}_\ell = \phi$ for $\ell = 1$, $1 \le j \le q(x)$, and for $\ell = 2$, $m+1 \le j \le q(x)$.

Define $Q_+{}^j{}_{\ell+1}$ for all $(j,\ell) \in \beta(f)$, that is,

(3.6)
$$Q_+{}^j{}_{\ell+1} = \begin{cases} Q^j{}_\ell & \text{if } x(j,\ell) = 0, \\ Q^j{}_\ell \cup \{n+1\} & \text{if } x(j,\ell) = 1. \end{cases}$$

Let $\theta^1 = \theta^0 \cap \beta(f)$. Define for all $(j',\ell') \in f(\theta^1)$,

(3.7) $\quad Q_+{}^{j'}{}_{\ell'} = Q_+{}^j{}_\ell - P_+{}^{j'}{}_{\ell'}$

where (j,ℓ) is defined by (3.1) and $Q_+{}^j{}_\ell$ is already defined by (3.6). Let $\theta^2 = \theta^0 \cap f(\theta^1)$. $\theta^1 \cap \theta^2 = \phi$ since $\theta^1 \subset \beta(f) = \alpha \backslash f(\theta^0)$ and $\theta^2 \subset f(\theta^1) \subset f(\theta^0)$. Hence $f(\theta^1) \cap f(\theta^2) = \phi$ since f is one-to-one. Define $Q_+{}^{j'}{}_{\ell'}$ by (3.7), for all $(j',\ell') \in f(\theta^2)$. Let $\theta^3 = \theta^0 \cap f(\theta^2)$. $\theta^3 \cap \theta^2 = \phi$ since $\theta^3 \subset f(\theta^2)$, $\theta^2 \subset f(\theta^1)$, and $f(\theta^1) \cap f(\theta^2) = \phi$. $\theta^3 \cap \theta^1 = \phi$ since $\theta^3 \subset f(\theta^2) \subset f(\theta^0)$ and $\theta^1 \subset \beta(f) = \alpha \backslash f(\theta^0)$. Hence θ^1, θ^2 and θ^3 are mutually disjoint. $f(\theta^1)$, $f(\theta^2)$ and $f(\theta^3)$ are also mutually disjoint. Continue with this operation and the same argument until it occurs that

(3.8) $\theta^r \neq \phi$ and $\theta^{r+1} = \phi$.

$\theta^1, \ldots, \theta^r$ are mutually disjoint. Also $f(\theta^1), \ldots, f(\theta^r)$ are. Let $\delta \equiv \theta^0 \setminus (\theta^1 \cup \ldots \cup \theta^r)$. $\delta \subset \theta^0 \setminus \theta^1 \subset f(\theta^0)$. Hence $\delta \subset \theta^0 \cap f(\theta^0) = \theta^0 \cap (f(\theta^1) \cup \ldots \cup f(\theta^r) \cup f(\delta)) = (\theta^0 \cap f(\theta^1)) \cup \ldots \cup (\theta^0 \cap f(\theta^r)) \cup (\theta^0 \cap f(\delta)) = \theta^2 \cup \ldots \cup \theta^r \cup (\theta^0 \cap f(\delta))$. This and the definition of δ imply $\delta \subset \theta^0 \cap f(\delta)$. But $||\delta|| = ||f(\delta)||$ since f is one-to-one. Hence $\delta = f(\delta)$. But this is possible only if $\delta = \phi$ because of the definition of f and since δ is a finite set. Consequently we have:

(3.9) $\theta^0 = \theta^1 \cup \ldots \cup \theta^r$ (disjoint sum), and
$\alpha = \beta(f) \cup f(\theta^1) \cup \ldots \cup f(\theta^r)$ (disjoint sum).

Thus, for each $(j, \ell) \in \alpha$, $Q_+{}^j{}_\ell$ has been defined and

(3.10) $Q_+{}^j{}_{\ell+1}$ = either $Q^j{}_\ell$ or $Q^j{}_\ell \cup \{n+1\}$

by (3.1),(3.4),(3.5),(3.6),(3.7) and (3.8).

Let $S^j(f,x) = (P_+{}^j, Q_+{}^j(x))$ for $j = 1, \ldots, q(x)$. It follows that $S^j(f,x)$ is a signed partition on $N \cup \{n+1\}$, from the definitions of $P_+{}^j$ and $Q_+{}^j(x)$ i.e., (3.4), (3.5), (3.6), (3.7) and (3.10), and from the fact that S^j is a signed partition on N.

Proposition 5. Let

$$\mu_0 \equiv \sum_{j=1}^{m} {}^*v(S^j(f,x)) \;+^*\; \sum_{j=m+1}^{q(x)} {}^*v(S^j(f,x)).$$

Negative coefficients in the right hand side vanish.

By Proposition 5, $S^1(f,x), \ldots, S^{q(x)}(f,x)$ define a balanced set, $B(f,x)$, on $N \cup \{n+1\}$. Let $M_+ \equiv (m-m(x), m_1, \ldots, m_p) = (m-m(x), M)$ when $m > m(x)$ and let $M_{++} \equiv (m_1, \ldots, m_p, m(x)-m) = (M, m(x)-m)$ when $m \leq m(x)$. Then $\mu(B(f,x); M_+) = \mu_0$ when $m > m(x)$, and $\mu(B(f,x); M_{++}) = \mu_0$ when $m \leq m(x)$.

Theorem 6. Let B be a balanced set on N. Suppose B has two representations, say by $(S^1, \ldots, S^m)$ and by $(S'^1, \ldots, S'^m)$. Then

(3.11) $\{B\,(f,x)$: all f and all x from $(S^1, \ldots, S^m)\}$
$= \{B\,(f',x')$: all f' and all x' from $(S'^1, \ldots, S'^m)\}$.

4. Epsilon Cores

In this section we define a special epsilon in order to extend Theorem 1 in Section 1. Then we examine properties of the epsilon-shift. Let for a game $\Gamma = (N,v)$ and a condition α,

$$h^{\alpha}(\Gamma) \equiv \sup\{\langle w;v\rangle : w \in W^{\alpha}\}.$$

Then by (2.1) Γ satisfies a condition α if and only if $Nv(N) \geq h^{\alpha}(\Gamma)$. $h^{\alpha}(\Gamma)$ is, in a sense, an average of worthes of proper coalitions. For any Γ, $h^{\alpha}(\Gamma)$ has a value which is finite if $\alpha = b$ or $\alpha = e$ [9], since W^{α} is bounded. Let $\Gamma = (N,v)$ be a game. For conditions α and b, define

(4.1) $\varepsilon(\alpha) \equiv \varepsilon(\alpha,\Gamma) \equiv \sup\{[\langle w;v\rangle - h^{\alpha}(\Gamma)]/\underline{w} : w \in W^{b}\}$.

In (4.1) "sup" in the right hand side can be replaced by "max" if $h^{\alpha}(\Gamma)$ is finite, since W^b is compact and convex, and since $[\langle w;v\rangle - h^{\alpha}(\Gamma)]/\underline{w}$ is continuous in w and $\underline{w} > 0$[10]. Indeed $\varepsilon(\alpha)$ is the maximum of a finite number of quantities which are attained at extreme points of W^b. The following is an extension of Theorem 1.

Proposition 7. A game $\Gamma = (N,v)$ satisfies a condition α if and only if $h^{\alpha}(\Gamma)$ is finite and $C_{\varepsilon(\alpha)}(\Gamma) \neq \phi$.

Remark 8. $W^{\alpha} \supset W^{\alpha'} \Longleftrightarrow \varepsilon(\alpha,\Gamma) \leq \varepsilon(\alpha',\Gamma)$ for all Γ.
$\Longleftrightarrow C_{\varepsilon(\alpha)}(\Gamma) \subset C_{\varepsilon(\alpha')}(\Gamma)$ for all Γ,

where α and α' are conditions.

Proposition 9. Let $\Gamma = (N,v)$ be a game. Then

$$\varepsilon(\alpha,\Gamma) = \mathrm{Min}_{x(N)=h^{\alpha}(\Gamma)} \mathrm{Max}\{Nv(S) - x(S) : S \neq \phi, N\}.$$

Let $\Gamma = (N,v)$ be a game. For a real number ε, the ε-*shifted* game $\Gamma_{\varepsilon} \equiv (N,v_{\varepsilon})$ is

$$v_{\varepsilon}(S) = \begin{cases} v(N) & \text{if } S = N, \text{ and} \\ v(S) - \varepsilon & \text{if } S \neq \phi, N. \end{cases}$$

In particular, for a condition α, we write the $\varepsilon(\alpha)$-shift as ψ^{α}, that is, for a game $\Gamma = (N,v)$, $\psi^{\alpha}(\Gamma) \equiv (N,v_{\varepsilon(\alpha)})$. It is easy to see $C_{\varepsilon}(\Gamma) = C(\Gamma_{\varepsilon})$ and $X(\psi^{\alpha}(\Gamma)) = X(\Gamma)$ for all Γ. Let $G(\alpha) \equiv \{\Gamma = (N,v) : \Gamma \text{ is a game and } h^{\alpha}(\Gamma) < +\infty\}$. ψ^{α} is a mapping from $G(\alpha)$ to $G \equiv \{\Gamma = (N,v) : \Gamma \text{ is a game}\}$.

Proposition 10. Let $\Gamma^1 = (N,v^1)$ and $\Gamma^2 = (N,v^2)$ be games such that $\Gamma^1, \Gamma^2 \in G(\alpha)$. Then
(i) $\psi^{\alpha}(\Gamma^1) = \Gamma^2 \Rightarrow h^{\alpha}(\Gamma^1) + n\varepsilon(\alpha,\Gamma^1) = h^{b}(\Gamma^2)$.
(ii) $\psi^{\alpha}(\Gamma^1) = \psi^{\alpha}(\Gamma^2) \Rightarrow h^{\alpha}(\Gamma^1) + n\varepsilon(\alpha,\Gamma^1) = h^{\alpha}(\Gamma^2) + n\varepsilon(\alpha,\Gamma^2)$.
(iii) $\psi^{\alpha}(\Gamma^1) = \psi^{\alpha}(\Gamma^2) \Rightarrow v^1 = v^2_a$ for some real number a.

Theorem 11. Assume W^{α} is convex, it has a finite number of extreme points, and $\underline{w} > 0$ for all $w \in W^{\alpha}$. Then
(i) $\psi^{\alpha} : G(\alpha) \to G$ is one-to-one.
(ii) Moreover, if W^{α} is bounded, then ψ^{α} is also onto and continuous.
(iii) The inverse of ψ^{α} is given by : For $\Gamma = (N,v)$, there exists $w\# \in W^{b}$ such that $\langle w\#;v\rangle = h^{b}(\Gamma)$. Choose ε so that $h^{\alpha}(\Gamma_{-\varepsilon}) + n\varepsilon = \langle w\#;v\rangle$. Then $(\psi^{\alpha})^{-1}(\Gamma) = \Gamma' = (N,u)$, where $u(S) = v(S) + \varepsilon$ for all $S \neq \phi, N$.
(iv) If $\Gamma = (N,v)$ is additive, i.e., $Nv(S) = 0$ for all $S \subset N$, then $\Psi^{\alpha}(\Gamma) = \Gamma$.

If W^{α} is not bounded then Ψ^{α} is not necessarily onto. To see this, suppose α = "*c* ", i.e., "convex." W^{c} is not bounded. Let $\Gamma = (N,v)$. Let $N = \{1,2,3\}$, $v(N) = 1$, $v(\{23\}) = v(\{13\}) = v(\{12\}) = -1$, and $v(\{i\}) = 0$ for $i = 1,2,3$.

Assume $\psi^c(\Gamma') = \Gamma$ and $\Gamma' = (N,u)$. Then $u(\{ij\}) - \varepsilon(c, \Gamma') = -1$ and $u(\{i\}) - \varepsilon(c, \Gamma') = 0$. Hence Γ' is symmetric. Let $a = \varepsilon(c, \Gamma')$. $b_2 \equiv u(\{ij\}) = a-1$, $b_1 \equiv u(\{i\}) = a$.

$$h^c(\Gamma') = \sup\{(b_2 - 2b_1)[w_{23}+w_{13}+w_{12}] : w \in W^c\}$$
$$= \begin{cases} 2b_2 - 4b_1 & \text{if } b_2 \geq 2b_1, \text{ and} \\ +\infty & \text{otherwise.} \end{cases}$$

Hence, if $b_2 \geq 2b_1$, $a = \varepsilon(c, \Gamma') = (2b_1 - b_2)/3$, which, combined with $b_2 = a-1$ and $b_1 = a$, implies $b_1 = 1/2$ and $b_2 = -1/2$. But this contradicts $b_2 \geq 2b_1$. Hence Ψ^c is not onto. Note that $C(\Gamma) \neq \phi$.

From Proposition 7 and Theorem 11, if W^α is bounded then Ψ^α is a transformation between the set of games with condition a and the set of balanced games, which is one-to-one and onto.

5. Remarks

(i) Let $\Gamma = (N,v)$ be a game. Define a half-space in W by $W(\Gamma) = \{w \in W : \mathcal{N}v(N) \geq \langle w;v \rangle\}$. Conversely, a half-space in W characterizes a game to some extent. That is, we have :

Proposition 12. Let $\Gamma = (N,v)$ and $\Gamma' = (N,v')$ be games. Then $W(\Gamma) = W(\Gamma')$ if and only if either
<u>(1)</u> Γ and Γ' are strategically equivalent, i.e., there exist a positive number k and real numbers a_i ($i \in N$) such that $v(S) = kv'(S) + a(S)$ for all $S \subset N$,
or
<u>(2)</u> Γ is additive and Γ' satisfies $\mathcal{N}v'(S) = 0$ for all $S \neq N$ and $\mathcal{N}v'(N) > 0$.

In the same way we easily have :

Proposition 13. Let $\Gamma = (N,v)$ be a game. $W(\Gamma) = W$ if and only if either <u>(1)</u> Γ is an additive game or <u>(2)</u> it satisfies $\mathcal{N}v(S) = 0$ for all $S \neq N$ and $\mathcal{N}v(N) > 0$. $W(\Gamma) = \phi$ if and only if $\mathcal{N}v(N) < 0$ and $\mathcal{N}v(S) = 0$ for all $S \neq N$.

(ii) The Shapley value [6] and the nucleolus [5] are solution-concepts for characteristic-function games with sidepayments. They are invariant under the transformation Ψ^α. Thus Γ and $\Psi^\alpha(\Gamma)$ have the same Shapley value and the nucleolus.

(iii) With respect to the problem of existence of the core in the games without sidepayments, we can find some papers stating the generalizations of the K-K-M Theorem. [9] is the first which treated the subject on this line.

6. References:

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[1]They have given finer results than that in Theorem 1. See Theorem 2 at p.457 of Shapley [7].
[2]Bondareva [2] defined a generalized covering, which is slightly different from the definition of generalized partition
[3]See Shapley [7] for the definition of balancedness. (1.2) and (2.1) are examples of balanced inequalities.
[4]See Shapley [8].
[5]"Total balancedness" is a well-known example of a condition which has the totality. See Shapley/Shubik [10].
[6]See p.457 of Shapley[7].
[7]From this $Q_1 = \phi$, but we use Q_1 for convenience.
[8]It is well-known that a balanced set has at least one family of associated coefficients consisting of rational numbers. See Shapley [7].
[9]We assume v(S) is a finite number for every $S \subset N$.
[10]It is easy to see $\underline{w} \geq n/(n-1)$ if w is associated with a balanced set.