数理解析研究所講究録
第 727 巻 1990 年 42-52

Extended automorphic forms on the upper half plane

W. Casselman

Introduction

Formally,

$$\int_0^\infty x^s\, dx = \int_0^1 x^s\, dx + \int_1^\infty x^s\, dx.$$

The first integral on the right converges for $\mathrm{Re}(s) > -1$ and is then equal to $1/(s+1)$. The second converges for $\mathrm{Re}(s) < -1$ and is then equal to to $-1/(s+1)$. Hence, defining the integral by analytic continuation, we can write (at least if $s \neq -1$)

$$\int_0^\infty x^s\, dx = 0.$$

This paper will show how to justify this calculation and similar ones where the domain of integration is replaced by quotients of the upper half plane by arithmetic groups and the integrand by certain automorphic forms, and will explain by examples why being able to justify such calculations is useful. The first example will be a new calculation of the area of such a quotient, and the second will be a new derivation of the Maass-Selberg formula for the inner product of truncated Eisenstein series. All of the notions in this paper can be extended with essentially technical modifications to the case of arithmetic groups of arbitary rank, and these calculations generalize easily to give respectively the volume formula of Langlands (1966a) and Lai (1980) and the inner product formula of Langlands (1966b) and Arthur (1980) (see also Labesse-Langlands (1983) and Arthur (1982)).

When I described some of the results of this paper to Jacquet several years ago, he pointed out similarities with the ideas of Zagier (1982), and the current formulation has been strongly influenced by that paper. It provides a more automatic method of justifying Zagier's calculations, and in such a way that extension to groups of higher rank is straightforward. It is worthy of remark that one of Zagier's topics is the Selberg trace formula, which is not covered here at all, but I would like to think that eventually a serious connection may be found.

It is also perhaps worth remarking that the techniques of this paper extend with mild modification to reductive groups over local fields. The main application to local representation theory is an inner product formula for truncated matrix coefficients, generalizing the formula of Waldspurger (1987), and it turns out that this can be used to give a direct construction of the Plancherel measure.

1. The multiplicative group

The simplest reductive arithmetic quotient is the multiplicative group of positive real numbers. Define the **Schwartz space** $\mathcal{S}(0,\infty)$ to be the space of all functions in the Schwartz space of $\mathbb{R}$ which vanish identically on $(-\infty,0)$. In other words a smooth function f on the open interval $(0,\infty)$ lies in this space if and only if it and all the derivatives d^nf/dx^n vanish at infinity and at 0 more rapidly than any power of x. This is equivalent to the same condition on the derivatives D^nf where D is the multiplicatively invariant derivative xd/dx.

In this paper I define the **extended Schwartz space** $\mathcal{S}[0,\infty)$ to be that of all smooth functions f on $(0,\infty)$ obtained by restricting to $(0,\infty)$ smooth functions on all of $\mathbb{R}$ which lie in the Schwartz space of $\mathbb{R}$ and whose Taylor series at 0 comprises just a constant term. This definition is somewhat arbitrary, but in the context of arithmetic quotients seems to balance nicely several ultimate requirements. For representations of reductive groups over $\mathbb{R}$, the correct definition would be simply the restrictions of all functions in the Schwartz space of $\mathbb{R}$. The difference arises in the difference between the asymptotic behaviour of automorphic forms and of matrix coefficients, and in the difference in ease with which one can use compactifications of the spaces at hand.

Define the spaces of **tempered distributions** $A(0,\infty)$ and **extended distributions** $A[0,\infty)$ to be the respective duals. The multiplicative group $\mathbb{R}^{\mathrm{pos}}$ acts on these spaces by the right regular representation.

The canonical inclusion gives a short exact sequence of $\mathbb{R}^{\mathrm{pos}}$-modules

$$0 \to \mathcal{S}(0,\infty) \to \mathcal{S}[0,\infty) \to \mathbb{C} \to 0$$

where the the third term is the trivial representation, and dually

$$0 \to \mathbb{C} \to A[0,\infty) \to A(0,\infty) \to 0. \tag{1.1}$$

Let $\mathbb{C}[s]$ be the one-dimensional module on which $\mathbb{R}^{\mathrm{pos}}$ acts by the character x^s, and on which the differential operator D acts as multiplication by s. The integration formula

$$\langle x^s, f\rangle = \int_0^\infty f(x)x^s\,dx/x$$

corresponds to an $\mathbb{R}^{\mathrm{pos}}$-injection of $\mathbb{C}[s]$ into $A(0,\infty)$. As a functional as well as a function it thus satisfies the equation $Dx^s = sx^s$.

1.1. Proposition. *For $s \neq 0$ there exists a unique eigenvector of D in $A[0,\infty)$ extending the functional x^s on $\mathcal{S}(0,\infty)$.*

I will call the extension $\mathfrak{e}(x^s)$. Zagier calls it the **renormalization** of x^s.

Proof. Any short exact sequence of $C[D]$-modules

$$0 \to A \to B \to C \to 0$$

gives a long exact sequence

$$\begin{aligned} 0 \to \mathrm{Ker}_A(D-s) \to &\mathrm{Ker}_B(D-s) \to \mathrm{Ker}_C(D-s) \\ &\to A/(D-s)A \to B/(D-s)B \to C/(D-s)C \to 0. \end{aligned}$$

Hence if $D-s$ is invertible on C with itself, inclusion induces an isomorphism of $A/(D-s)A$ with $B/(D-s)B$.

Apply this argument to the sequence (1.1). If s is not 0 then $D-s$ is an isomorphism on $\mathbb{C}$, so that restriction induces an isomorphism between the subspaces of elements of $A[0,\infty)$ and $A(0,\infty)$ annihilated by $D-s$. □

Another way of formulating the main point of the argument is to say that the **spectrum** of the quotient $\mathcal{S}[0,\infty)/\mathcal{S}(0,\infty)$ (with respect to the the operator D) is exactly $\{0\}$.

More explicitly, since the derivative of any f in $\mathcal{S}[0,\infty)$ lies in $\mathcal{S}(0,\infty)$, for every $s \neq 0$ integration by parts suggests the definition

$$\langle \mathfrak{e}(x^s), f\rangle = -\frac{1}{s}\left\langle x^{s+1}, df/dx\right\rangle$$

This definition of the extended functional $\mathfrak{e}(x^s)$ (a very simple case of Hadamard's *partie finie*) is easily seen to agree with the previous one. It shows that the definition of the extension may be characterized as by analytic continuation.

More generally, if M is any module over the polynomial algebra $\mathbb{C}[D]$ on which multiplication by D is bijective (say on which $D-s$ acts nilpotently for some $s \neq 0$), then the proof of Proposition 1.1 shows that any D-covariant map from M to $A(0,\infty)$ possesses a unique lifting to $A[0,\infty)$. Corresponding to the product formula

$$Dx^s \log^n(x) = sx^s \log(x) + nx^s \log^{n-1}(x)$$

or again integration by parts, we define the extension of the distribution associated to $x^s \log^n(x)$ by induction on n:

$$\langle \mathfrak{e}(x^s \log^n(x)), f\rangle = -\frac{1}{s}\int_0^\infty x^s \log^n(x) df/dx\, dx - \frac{n}{s} < \mathfrak{e}(x^s \log^{n-1}(x)), f > .$$

For every s in $\mathbb{C}$ define $A_s(0,\infty)$ to be the subspace of distributions of the form $x^s P(\log^x)$ where P is a polynomial. Then in summary:

1.2. Proposition. *For $s \neq 0$ every element in $A_s(0\infty)$ possesses a canonical lifting to $A[0,\infty)$. Morre precisely, if P is a polynomial of degree n then $\mathfrak{e}(x^s P(\log x)$ varies meromorphically with s, holomorphically in the region $\mathbb{C} - \{0\}$, and with a pole of order n at 0. In particular the residue of x^s at 0 is δ_0.*

We are not quite to the point where we can integrate x^s over all of $(0,\infty)$, but close. Define now third and fourth Schwartz spaces $\mathcal{S}(0,\infty]$ and $\mathcal{S}[0,\infty]$ to be respectively the smooth functions f on $(0,\infty)$ such that $f(1/x)$ lies in $\mathcal{S}[0,\infty)$; the sums of functions in $\mathcal{S}[0,\infty)$ and $\mathcal{S}(0,\infty]$. Define $A(0,\infty]$, $A[0,\infty]$ to be their duals. Since the constant functions are elements of $\mathcal{S}[0,\infty]$, any element of $A[0,\infty]$ can in effect be integrated over the whole interval $(0,\infty)$. All that remains is to show:

1.3. Proposition. *For each $s \neq 0$ every element Φ of $A(0,\infty)$ possesses a canonical extension $\mathfrak{e}(\Phi)$ to $A[0,\infty]$. This functional varies meromorphically with s and has a single pole at 0.*

Proof. Not much different from that of Proposition 1.1. Explicitly one can define the extension locally near 0 and ∞, by writing an element of $\mathcal{S}[0,\infty]$ as a sum of two parts. □

If δ_∞ is the functional on $\mathcal{S}[0,\infty]$ which takes f to $f(\infty)$:

1.4. Proposition. *The residue of the extension of x^s at $s = 0$ is equal to $\delta_0 - \delta_\infty$.*

Proof. I leave this as an exercise. □

1.5. Corollary. *For $s \neq 0$,*

$$\langle \mathfrak{e}(x^s \log^n(x)), 1\rangle \left(\textit{formally} \int_0^\infty x^s \log^n(x)\, dx/x\right) = 0.$$

Proof. Look at the case $n = 1$: by definition of the dual representation

$$\begin{aligned}\langle Dx^s, 1\rangle &= s\,\langle x^s, 1\rangle \\ &= \langle x^s, D1\rangle = 0. \quad \square\end{aligned}$$

It is useful to keep in mind that multiplication makes the space $A[0,\infty]$ into a module over $\mathcal{S}[0,\infty]$:

$$\langle f\Phi, \varphi\rangle = \langle \Phi, f\varphi\rangle .$$

Similarly of course if s, t, $s+t$ are all non-zero then the product of any two elements F_t and F_s in A_s and A_t, which lies in A_{s+t}, as well as F_s and F_t, have canonical extensions. The product rule for differentiation holds in all these cases. Hence:

1.6. Proposition. *For any two functions F_s and F_t in $A_s[0,\infty)$ and $A_t[0,\infty)$ (s, t, $s+t$ not 0) we have*

$$\langle DF_s \cdot F_t, 1\rangle = -\langle F_s \cdot DF_t, 1\rangle ,$$

and more generally

$$\langle LF_s \cdot F_t, 1\rangle = \langle F_s \cdot L^*F_t, 1\rangle$$

for any polynomial L in D, where L^* is the differential operator adjoint to L.

It is also useful to keep in mind that if χ is the characteristic function of a half-line $(0,T]$ or $[T,\infty)$, then $\chi(x)x^s$ can be interpreted canonically as an element of $A[0,\infty]$: write f in $\mathcal{S}[0,\infty]$ as a sum of elements f_1, f_2 in $\mathcal{S}[0,\infty]$, one with support on the half-line, the other eventually identically vanishing on the half-line. Then the definition

$$\langle \mathfrak{e}(\chi(x)x^s), f\rangle := \langle \mathfrak{e}(x^s), f_1\rangle + \langle \chi(x)x^s, f_2\rangle$$

is independent of the particular choice of the f_i. It is easy to see that (writing a little loosely)

$$\int_0^T x^s\,dx/x = T^s/s, \qquad \int_T^\infty x^s\,dx = -T^s/s$$

for all s where $\mathfrak{e}(x^s)$ is defined.

2. The upper half plane

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ such that (a) $\Gamma\backslash\mathfrak{H}$ has finite volume and (b) Γ has one cusp. The second condition is for convenience. Let P be the subgroup of upper triangular matrices in $\mathrm{SL}_2(\mathbb{R})$, N its subgroup of unipotent matrices. For further convenience I will assume that the cusp is ∞ and that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generates $\Gamma \cap N$. The best known example would be $\mathrm{SL}_2(\mathbb{Z})$.

Choose on $\mathfrak{H}$ the $\mathrm{SL}_2(\mathbb{R})$-invariant metric

$$(dx^2 + dy^2)/y^2$$

with associated volume form $y^{-2}\, dx\, dy$ and associated Laplacian

$$\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2).$$

I recall that the Schwartz space $\mathcal{S}(\Gamma\backslash\mathfrak{H})$ is the space of smooth functions f on $\Gamma\backslash\mathfrak{H}$ such that f and all its derivatives $\partial^n f/\partial^m x\, \partial^{n-m} y$ decrease more rapidly than any power of y as y goes to infinity. Define the **extended Schwartz space** $\mathcal{S}_{\mathrm{Ext}}(\Gamma\backslash\mathfrak{H})$ of the quotient to be the space of all smooth functions f on $\Gamma\backslash\mathfrak{H}$ which may be expressed as the sum of a function in $\mathcal{S}(\Gamma\backslash\mathfrak{H})$ and a function of y alone which lies in $\mathcal{S}(0,\infty]$. Define $A(\Gamma\backslash\mathfrak{H})$, $A_{\mathrm{Ext}}(\Gamma\backslash\mathfrak{H})$ to be the respective dual spaces.

Of course since functions in $\mathcal{S}_{\mathrm{Ext}}(\Gamma\backslash\mathfrak{H})$ are bounded, there is a canonical embedding of $L^1(\Gamma\backslash\mathfrak{H})$, and of its subspace $L^2(\Gamma\backslash\mathfrak{H})$, into $A_{\mathrm{Ext}}(\Gamma\backslash\mathfrak{H})$.

We have a short exact sequence of Δ-spaces

$$0 \to \mathcal{S}(\Gamma\backslash\mathfrak{H}) \to \mathcal{S}_{\mathrm{Ext}}(\Gamma\backslash\mathfrak{H}) \to \mathbb{C} \to 0.$$

Recall that the **Eisenstein series** E_s for s in $\mathbb{C}$ with $\mathrm{Re}(s) > 1$ is defined by the convergent series

$$E_s(z) = \sum_{\Gamma\cap P\backslash\Gamma} y(\gamma z)^s.$$

If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ then this is the Maass function

$$\sum_{(c,d)=1,c>0} \frac{y^s}{|cz+d|^{2s}}.$$

The function E_s is real-analytic on $\Gamma\backslash\mathfrak{H}$, and an eigenfunction of Δ with eigenvalue $s(s-1)$. It continues meromorphically as a function of the parameter s to all of $\mathbb{C}$; for all s where defined satisfies the equation

$$\Delta E_s = s(1-s)E_s;$$

and near infinity is the sum of a Schwartz function and its constant term $y^s + c(s)y^{1-s}$, where $c(s)$ is a meromorphic scalar function on $\mathbb{C}$. Furthermore E_s satisfies the functional equation

$$E_s = c(s)E_{1-s}.$$

If Γ is $\mathrm{SL}_2(\mathbb{Z})$ then

$$c(s) = \frac{\xi(2s-1)}{\xi(2s)}, \qquad \left(\xi(s) = \pi^{s/2}\Gamma(s/2)\zeta(s)\right).$$

2.1. Proposition. *Any automorphic form Φ whose constant term contains no components of the form $y\log^n(y)$ has a canonical extension $\mathfrak{e}(\Phi)$ in $A_{\mathrm{Ext}}(\Gamma\backslash\mathfrak{H})$. In particular any E_s where E_s is defined and $s \neq 0, 1$ has a unique extension which remains an eigenvector of Δ.*

Proof. One could use reasoning similar to that in the proof of Proposition 1.1. It is better to be more explicit. Given f in $\mathcal{S}_{\mathrm{Ext}}(\Gamma\backslash\mathfrak{H})$, let f_∞ be the function $\chi(y)f_0(y)$ made Γ-invariant, where $f_0(y)$ is the constant term of f, and $\chi(y)$ is a suitable smooth cut-off function, identically 1 for y large, identically 0 for y a little less large. Then $f - f_\infty$ lies in the Schwartz space and we set

$$\langle \mathfrak{e}(\Phi), f\rangle = \langle \Phi, f - f_\infty\rangle + \langle \varphi, f_\infty\rangle$$

where φ is the constant term of Φ. □

The poles of E_s in the right hand plane $\mathrm{Re}(s) \geq \frac{1}{2}$ are known to lie in the interval $(\frac{1}{2}, 1]$. The residue at 1 is a constant function (since it is harmonic and asymptotic to a constant, the residue $c^*(1)$ of $c(s)$ at 1). If Γ is $\mathrm{SL}_2(\mathbb{Z})$ this residue is the constant function $1/\xi(2)$. This residue is of course also its residue as a tempered distribution on $\Gamma\backslash\mathfrak{H}$. The extended distribution $\mathfrak{e}(E_s)$ has also a simple pole, but its residue is slightly more complicated. Note that at least since constants are square-integrable, they may be canonically identified with extended distributions.

2.2. Proposition. *The residue of* $\mathfrak{e}(E_s)$ *at* $s = 1$ *is* $c^*(1) - \delta_\infty$.

Since constants lie in $\mathcal{S}_{\mathrm{Ext}}(\Gamma\backslash\mathfrak{H})$, any extended distribution may be paired with constants, or in other words integrated over $\Gamma\backslash\mathfrak{H}$. Since the extension of E_s is still an eigenfunction for Δ, an argument entirely similar to that for the proof of Proposition 1.4 implies:

2.3. Lemma. *For non-integral* s *where* E_s *is defined*

$$\langle E_s, 1\rangle \ \Big(\text{formally} \int_{\Gamma\backslash\mathfrak{H}} E_s \, \frac{dx\,dy}{y^2}\Big) = 0.$$

Letting $s = 1$ and applying Proposition 2.2:

2.4. Proposition. *The volume of the quotient is*

$$\int_{\Gamma\backslash\mathfrak{H}} \frac{dx\,dy}{y^2} = \frac{1}{c^*(1)}.$$

The product of two Eisenstein series E_s and E_t has as asymptotic expansion near infinity the sum of four terms whose exponents are $s+t$, $2-s-t$, $1-s+t$, $1-t+s$. As long as s and t are not singular parameters and as long as none of these exponents is 0, the product E_sE_t can be extended canonically, hence integrated over all of the quotient.

2.5. Proposition. *Whenever the product* E_sE_t *may be canonically extended*

$$\langle E_sE_t, 1\rangle = \int_{\Gamma\backslash\mathfrak{H}} E_sE_t \, \frac{dx\,dy}{y^2} = 0.$$

Proof. Apply Proposition 1.6. □

Let T by a large positive number. Define the corresponding **truncation** $\Lambda^T(F)$ of any function F on $\Gamma\backslash\mathfrak{H}$ to be just F in the region $y \le T$ of the fundamental domain, but $F - F_0$ in the region $y > T$. Let $C^T(F)$ be the remainder, equal to 0 inside amnd F_0 outside. The decomposition

$$F = \Lambda^T(F) + C^T(F)$$

is orthogonal in practically any sense. If F is an automorphic form, then its truncation is rapidly decreasing near infinity. In particular the truncation of E_s will be square-integrable. Since truncation is an orthogonal projection, it may be therefore defined on the spaces of tempered distributions and extended tempered distributions by duality.

2.6. Corollary. (The Maass-Selberg formula) *For any values of s and t*

$$\langle \Lambda^T(E_s), \Lambda^T(E_t)\rangle = \frac{T^{s+t-1} - c(s)c(t)T^{1-s-t}}{s+t-1} + \frac{c(t)T^{s-t} - c(s)T^{t-s}}{s-t}.$$

Proof. For generic s and t the product of E_s and E_t may integrated over $\Gamma\backslash\mathfrak{H}$ and gives 0. But expressing E_s as $\Lambda^T(E_s) + C^T(E_s)$ we have

$$\langle \Lambda^T(E_s), \Lambda^T(E_t)\rangle + \langle C^T(E_s), C^T(E_t)\rangle = 0,$$

where the second term must be interpreted in the light of the remarks at the end of the previous section, that is to say by analytic continuation. Thus in the same spirit

$$\langle \Lambda^T(E_s), \Lambda^T(E_t)\rangle = \int_0^T (y^s + c(s)y^{1-s})(y^t + c(t)y^{1-t})\,\frac{dy}{y^2}$$

which gives the Proposition. □

References

1. J. Arthur, 'A trace formula for reductive groups II: Applications of a truncation operator', Comp. Math. **40** (1980), 87–121.

2. J. Arthur, 'On the inner product of truncated Eisenstein series', Duke Math. Jour. **49** (1982), 35–70.

3. J.-P. Labesse and R. P. Langlands, 'The morning seminar on the trace formula', preprint, Institute for Advanced Study, Princeton, 1983.

4. K. F. Lai, 'Tamagawa numbers of reductive algebraic groups', Comp. Math. **41** (1980), 153–188.

5. R. P. Langlands, 'The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups', pp. 143–148 in *Algebraic groups and discrete groups*, Proc. Symp. Pure Math. **IX**, edited by A. Borel and D. Mostow. Amer. Math. Soc., Providence, 1966.

6. R. P. Langlands, 'Eisenstein series', pp. 235–252 in *Algebraic groups and discrete groups*, Proc. Symp. Pure Math. **IX**, edited by A. Borel and D. Mostow. Amer. Math. Soc., Providence, 1966.

7. J-L. Waldspurger, 'À propos des intégrales orbitales pour $GL(n)$', Université Paris VII, 1987.

8. D. Zagier, 'The Rankin-Selberg method for automorphic functions which are not of rapid decay', Jour. Fac. Sci. Univ. Tokyo **28** (1982), 415–437.

Mathematics Institute
Faculty of Science
Tohoku University
Sendai 980, Japan

Electronic mail address:
Until August 31, 1990: `i8a1s1c@jpntohok.bitnet`
After then: `usercass@ubcmtsg.bitnet`