数理解析研究所講究録
第 727 巻 1990 年 140-155


# ある4次体の量指標のLに対応する保型形式について

名古屋市立保育短大 丹羽伸二 (Shinji Niwa)

In [5], H.Maass showed that the dimension of a space of generalized Whittaker functions satisfying certain system of differential equations on Siegel's upper half space $H_2$ of degree 2 is three. First of all, we shall investigate the structure of a space of generalized Whittaker functions which are eigen functions for the algebra of invariant differential operators on $H_2$. The theory of generalized Whittaker functions is discussed in Yamasita [12], [13], [14], [15] with full generality. We can get an outlook of the space of generalized Whittaker functions by using elementary calculus instead of representation theory of Lie groups. Generalized Whittaker functions naturally appear in the theory of indefinite theta function. We shall show commutation relations between the invariant differential operators on $H_2$ and those on the product $H_1 x H_1$ of two copies of the upper half plane $H_1$ with respect to a theta function. The relations are analogues of commutation relations for Hecke operators in [1], [17], [16] and are proved in some cases for Laplacian in [8], [2]. We essentially use the result in Nakajima [10] where the generators of the center of the universal enveloping algebra of Sp(2,$\mathbb{R}$) are explicitly given. By commutation relations we can construct an automorphic form F on $H_2$ corresponding to L-function with Grössencharacter of a certain biquadratic field. Generalized whittaker functions investigated appear in the Fourier expansion of F with respect to translations in $H_2$ and so we can define a "Fourier" coefficient which is a constant by ratio the Fourier coefficient to a generalized Whittaker function. A certain "Fourier" coefficient of F is given in 2.

1. We denote, as usual, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field, and the complex number field. We denote the algebra of invariant differential operators on $H_2$ by $\mathcal{D}$ and the center of the universal enveloping algebra by $\mathcal{C}$. Nakajima calculated generators of $\mathcal{D}$, $\mathcal{C}$ in [9], [10]. The generators of $\mathcal{D}$ are

$$\Delta_1 = \sum_{i,j=1}^{3} y_i y_j \partial_i \bar{\partial}_j - d(\partial_1\bar{\partial}_3 + \bar{\partial}_1\partial_3 - (1/2)\partial_2\bar{\partial}_2)$$

and

$$\begin{aligned}\Delta_2 = {} & d^2(\partial_1\partial_3 - (1/4)\partial_2{}^2)(\bar{\partial}_1\bar{\partial}_3 - (1/4)\bar{\partial}_2{}^2) \\ & +\sqrt{-1}(1/4)d(\sum_{i=1}^{3} y_i\partial_i)(\bar{\partial}_1\bar{\partial}_3 - (1/4)\bar{\partial}_2{}^2) \\ & +\sqrt{-1}(1/4)d(\sum_{i=1}^{3} y_i\bar{\partial}_i)(\partial_1\partial_3 - (1/4)\partial_2{}^2)\end{aligned}$$

$$+(1/16)d(\partial_1\bar{\partial}_3 + \bar{\partial}_1\partial_3 - (1/2)\partial_2\bar{\partial}_2)$$

where we put

$$z_i = x_i + \sqrt{-1}y_i \quad (1\le i\le 3),$$
$$\partial_i = \frac{\partial}{\partial z_i} = (1/2)\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right) \quad (1\le i\le 3),$$
$$\bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i} = (1/2)\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right) \quad (1\le i\le 3),$$

and $d=y_1y_3-y_2^2$ for $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}\in H_2$. For two complex numbers $d_1$, $d_2$, we consider the space $\mathcal{W}$ formed by functions $f(Z)=g(Y)$ of $Z=X+iY\in H_2$, satisfying $\Delta_1 f=d_1 f$ and $\Delta_2 f=d_2 f$ with some functions $g$ of $Y$, which we call generalized Whittaker functions. If $g(Y)e^{2\pi i \operatorname{tr} X}$ belongs to $\mathcal{W}$, $g$ has to satisfy

$$\text{(1.1)} \quad \frac{1}{8}\Big(4y_3y_2\frac{\partial^2}{\partial y_3\partial y_2} + 4y_2^2\frac{\partial^2}{\partial y_3\partial y_1} + 2y_3^2\frac{\partial^2}{\partial y_3^2} + 4y_2y_1\frac{\partial^2}{\partial y_2\partial y_1} + y_3y_1\frac{\partial^2}{\partial y_2^2} + y_2^2\frac{\partial^2}{\partial y_2^2} + 2y_1^2\frac{\partial^2}{\partial y_1^2} - 8\pi^2y_3^2 - 16\pi^2y_2^2 - 8\pi^2y_1^2\Big)g=d_1g$$

and

$$\text{(1.2)} \quad -\frac{d}{256}\Big(2(16d\pi^2+1)\frac{\partial^2}{\partial y_2^2} - 32(8d\pi^2-1)\pi^2 + d\Big(8\frac{\partial^4}{\partial y_1\partial y_2^2\partial y_3} - 16\frac{\partial^4}{\partial y_1^2\partial y_3^2} + 64\pi^2\frac{\partial^2}{\partial y_1^2} - \frac{\partial^4}{\partial y_2^4} + 64\pi^2\frac{\partial^2}{\partial y_3^2}\Big)$$

$$+ 16y_2 \frac{\partial^3}{\partial y_1 \partial y_2 \partial y_3} - 4y_1 \frac{\partial^3}{\partial y_1 \partial y_2{}^2} + 16y_3 \frac{\partial^3}{\partial y_1 \partial y_3{}^2}$$
$$- 8 \frac{\partial^2}{\partial y_1 \partial y_3} + 16y_1 \frac{\partial^3}{\partial y_1{}^2 \partial y_3} - 64\pi^2 y_3 \frac{\partial}{\partial y_1}$$
$$- 4y_2 \frac{\partial^3}{\partial y_2{}^3} - 4y_3 \frac{\partial^3}{\partial y_2{}^2 \partial y_3} + 64\pi^2 y_2 \frac{\partial}{\partial y_2}$$
$$- 64\pi^2 y_1 \frac{\partial}{\partial y_1}\Big) g = d_2 g$$

with $d=y_1y_3-y_2{}^2$. Put

$$(1.3) \qquad Y=\begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix}=\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$
$$g(Y)=h(t_1,t_2,\theta).$$

Then h is periodic with respect to $\theta$, so that h has Fourier expansion

$$(1.4) \qquad h(t_1,t_2,\theta)=\sum_{n\in\mathbb{Z}} B_n(t_1,t_2)e^{ni\theta}$$

where $B_n(t_1,t_2)$ is a solution of differential equations

$$(1.5) \qquad -4^{-1}(t_1-t_2)^{-2}\Big(4(t_1{}^2+t_2{}^2)(t_1-t_2)^2\pi^2+2n^2t_1t_2$$
$$-(t_1-t_2)^2(t_1{}^2\frac{\partial^2}{\partial t_1{}^2}+t_2{}^2\frac{\partial^2}{\partial t_2{}^2})$$
$$-t_1t_2(t_1-t_2)(\frac{\partial}{\partial t_1}-\frac{\partial}{\partial t_2})\Big)B=d_1B$$

and

$$(1.6)\qquad \frac{t_1t_2}{64(t_1-t_2)^4}\Big(2\Big(16(t_1-t_2)^2\pi^2t_1t_2+5t_1^2-18t_1t_2+5t_2^2\Big)n^2$$
$$+8(8\pi^2t_1t_2-1)(t_1-t_2)^4\pi^2+4n^4t_1t_2$$
$$-(t_1-t_2)\Big(4(2t_1-t_2)n^2t_2-16(t_1-t_2)^2\pi^2t_1^2-$$
$$-3t_1^2+8t_1t_2-3t_2^2\Big)\frac{\partial}{\partial t_2}$$
$$-(t_1-t_2)^2t_2\Big(16(t_1-t_2)^2\pi^2t_1+3t_1-2t_2\Big)\frac{\partial^2}{\partial t_2^2}$$
$$-(t_1-t_2)\Big(16(t_1-t_2)^2\pi^2t_2^2+4(t_1-2t_2)n^2t_1+3t_1^2$$
$$-8t_1t_2+3t_2^2\Big)\frac{\partial}{\partial t_1}$$
$$-t_1(t_1-t_2)^2\Big(16(t_1-t_2)^2\pi^2t_2^2-2t_1+3t_2\Big)\frac{\partial^2}{\partial t_1^2}$$
$$+4t_2^2(t_1-t_2)^3\frac{\partial^3}{\partial t_1\partial t_2^2}-4t_1^2(t_1-t_2)^3\frac{\partial^3}{\partial y_1^2\partial y_2}$$
$$+2(4n^2+1)t_1t_2(t_1-t_2)^2\frac{\partial^2}{\partial t_1\partial t_2}$$
$$+4t_1t_2(t_1-t_2)^4\frac{\partial^4}{\partial t_1^2\partial t_2^2}\Big)B_n=d_2B_n.$$

Now we introduce variables $x=t_1-t_2$, $y=t_1+t_2$ and put

$$(1.7)\qquad B_n(t_1,t_2)=C_n(x,y)$$

where $y>0$, $-y<x<y$. Then the differential equations for $B_n$ yield the condition for $C_n$

$$(1.8.1)\quad -\frac{1}{8x^2}\Big(4\pi^2x^2(x^2+y^2)+(y^2-x^2)n^2-x^2(x^2+y^2)\frac{\partial^2}{\partial y^2}-x^2(x^2+y^2)\frac{\partial^2}{\partial x^2}$$
$$-x(y^2-x^2)\frac{\partial}{\partial x}-4x^3y\frac{\partial^2}{\partial x\partial y}\Big)C_n=d_1C_n,$$

$$(1.8.2)\quad \frac{(y^2-x^2)}{256x^4}\Big((n^4+16\pi^4x^4)(y^2-x^2)+2(4\pi^2x^2y^2-4\pi^2x^4-2y^2+7x^2)n^2$$
$$-8\pi^2x^4+4x^3y(x\frac{\partial^3}{\partial y\partial x^2}+\frac{\partial^2}{\partial x\partial y}-x\frac{\partial^3}{\partial y^3})$$
$$+x^4(y^2-x^2)(\frac{\partial^4}{\partial y^4}-2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial x^4})$$
$$-2x^3(y^2+x^2)(\frac{\partial^3}{\partial x\partial y^2}-\frac{\partial^3}{\partial x^3})$$
$$-x^2(2n^2y^2-2n^2x^2+8\pi^2y^2x^2-8\pi^2x^4+y^2-3x^2)\frac{\partial^2}{\partial x^2}$$
$$+x(2n^2y^2-6n^2x^2-8\pi^2x^2y^2-8\pi^2x^4+y^2-7x^2)\frac{\partial}{\partial x}$$
$$+2x^2(n^2y^2-n^2x^2-4\pi^2x^2y^2+4\pi^2x^4+x^2)\frac{\partial^2}{\partial y^2}$$
$$-4x^2y(n^2+4\pi^2x^2)\frac{\partial}{\partial y}\Big)C_n=d_2C_n.$$

Put $C_n(x,y)=(x^2-y^2)\sum_{l=-\infty}^{\infty} a_l(y)x^l$ where $a_l(y)=0$ if $l\leq m$ with an integer $m\geq 0$ and $a_m(y)\neq 0$. Then the equation (1.8.1) implies

$$(1.9.1)\quad \frac{1}{8}\Big((l^2+2l+n^2-4\pi^2y^2-2)a_l+((l+2)^2-n^2)y^2a_{l+2}-4\pi^2a_{l-2}$$
$$+a''_{l-2}+y^2a''_l+(4l+1)ya'_l\Big)=d_1a_l,$$

especially $(m^2-n^2)y^2a_m=0$, $((m+1)^2-n^2)y^2a_{m+1}=0$ which result $m=n$, $a_{m+1}=0$ and therefore $a_l=0$ unless $l\equiv m \bmod 2$. The equation (1.8.2) implies

$$(1.9.2)\quad -\frac{1}{256}\Big(2(l+n+2)(l-n+2)*y^3\ (ya''_{l+2}+2a'_{l+2})$$
$$-(l^4-2l^2n^2+16l^2\pi^2y^2-6l^2+16l\pi^2y^2$$
$$-16l+n^4-16n^2\pi^2y^2-10n^2+16\pi^4y^4+40\pi^2y^2+12)a_l$$
$$-2(2l^2+2l-2n^2-4\pi^2y^2-1)y^2a''_l$$
$$+2(l^2+2l-n^2+4\pi^2\ y^2+7)$$

$$
\begin{aligned}
&(l + n + 2)(l - n + 2)y^2 a_{l+2} \\
&+ 8(l^2 - 2l - n^2 + 4\pi^2 y^2 - 3)\pi^2 a_{l-2} \\
&+ 2(l^2 - 2l - n^2 - 8\pi^2 y^2 + 7)a''_{l-2} \\
&- 4(l^2 - 8l - n^2 - 4\pi^2 y^2 - 8)y a'_l \\
&- (l + n + 4)(l + n + 2)(l - n + 4)(l - n + 2)y^4 a_{l+4} \\
&- a^{(4)}_{l-4} + 8\pi^2 a''_{l-4} + 2y^2 a^{(4)}_{l-2} + 4y a^{(3)}_{l-2} - 16\pi^2 y a'_{l-2} \\
&- y^4 a^{(4)}_l - 4y^3 a^{(3)}_l - 16\pi^4 a_{l-4}\Big) = d_2 a_l
\end{aligned}
$$

for all integers $l$. Since we especially have

$$
\begin{aligned}
(1.10)\qquad a_{n+2} &= \frac{1}{4(n+1)y^2}\Big(2(4d_1 - n^2 - n + 2\pi^2 y^2 + 1)a_n - 4(n+1)y a'_n - y^2 a''_n\Big), \\
a_{n+4} &= \frac{1}{8(n+1)y^2}\Big(2(4d_1 - n^2 - 3n + 2\pi^2 y^2 - 3)a_{n+2} - 4(n+3)y a'_{n+2} \\
&\qquad - y^2 a''_{n+2} - a''_n + 4\pi^2 a_n\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
(1.11)\qquad & 4(64d_2 + 4n^2 - 4n\pi^2 y^2 + 4n - 4\pi^4 y^4 - 10\pi^2 y^2 - 3)a_n \\
&+ 8y^2(2n + 4\pi^2 y^2 + 7)(n + 1)a_{n+2} + 16(2n + \pi^2 y^2 + 2)y a'_n \\
&- 2(2n - 4\pi^2 y^2 - 1)y^2 a''_n - 32(n + 2)(n + 1)a_{n+4} y^4 \\
&+ 8y^3(n + 1)(y a''_{n+2} + 2a'_{n+2}) - y^4 a^{(4)}_n - 4y^3 a^{(3)}_n = 0,
\end{aligned}
$$

by (1.9.1), (1.9.2), we obtain an ordinary differential equation

$$
\begin{aligned}
(1.12)\qquad & (8(n^2 + n + 1)d_1 - 16d_1^2 + 64d_2 - n^4 - 2n^3 + n^2 \\
&\quad + 2n + 8\pi^2 y^2)a_n \\
&+ 4(4(n + 1)d_1 - n^3 - 3n^2 - 2n + 4\pi^2 y^2)y a'_n
\end{aligned}
$$

$$+ 2(4d_1 - 3n^2 - 9n + 2\pi^2y^2 - 6)y^2 a_n''$$
$$- 4(n + 2)y^3 a_n^{(3)} - y^4 a_n^{(4)} = 0,$$

for $a_n$. It is more convenient to introduce parameters $\lambda_1$, $\lambda_2$ defined by

$$(1.13) \qquad d_1 = \frac{\lambda_1+\lambda_2-2}{8}, \quad d_2 = \frac{(\lambda_1-\lambda_2)^2}{256} - \frac{\lambda_1+\lambda_2}{32} + \frac{3}{64}$$

to describe the solutions of (1.12). With these $\lambda_1, \lambda_2$ (1.12) becomes

$$(1.14) \qquad - (\lambda_1\lambda_2 - n(n + 1)\lambda_1 - n(n + 1)\lambda_2$$
$$+ n^4 + 2n^3 + n^2 - 8\pi^2y^2) a_n$$
$$+ 2((\lambda_1 + \lambda_2)(n + 1) - 2n^3 - 6n^2 - 6n + 8\pi^2y^2 - 2)y a_n'$$
$$+ (\lambda_1 + \lambda_2 - 6n^2 - 18n + 4\pi^2y^2 - 14)y^2 a_n''$$
$$- 4(n + 2)y^3 a_n^{(3)} - y^4 a_n^{(4)} = 0.$$

For $\nu \in \mathbb{C}$, $m \in \mathbb{Z}$, denote the associated Legendre function of the first kind by $P_\nu^m(z)$ and that of the second kind by $Q_\nu^m(z)$ as usual. Then we have

$$(1.15) \qquad P_\nu^m(z) = \frac{\Gamma(\nu+m+1)}{\pi\,\Gamma(\nu+1)} \int_0^\pi \left(z + (z^2-1)^{1/2}\cos t\right)^\nu \cos mt\, dt,$$
$$Q_\nu^m(z) = (-1)^m \frac{\Gamma(\nu+m+1)}{2^{\nu+1}\Gamma(\nu+1)} (z^2-1)^{m/2} \int_{-1}^1 \frac{(1-t^2)^\nu}{(z-t)^{\nu+m+1}}\, dt$$

for z not on the real axis between 1 and $\infty$ where we assume

$w^{\mu}=e^{\mu\log w}$, $\log w=\log|w|+i\arg w$, $-\pi<\arg w<\pi$ for $w,\ \mu\in\mathbb{C}$. We put

$$(1.16)\qquad \begin{aligned} P_{\nu}^{m}(x)&=\lim_{y\to+0} e^{3m\pi i/2}\,P_{\nu}^{m}(x+iy)=\lim_{y\to+0} e^{m\pi i/2}\,P_{\nu}^{m}(x-iy),\\ Q_{\nu}^{m}(x)&=\frac{1}{2}\lim_{y\to+0}\left(e^{-m\pi i/2}\,Q_{\nu}^{m}(x+iy)-e^{m\pi i/2}\,Q_{\nu}^{m}(x-iy)\right)\end{aligned}$$

for $-1<x<1$. $P_{\nu}^{m}(z)$ and $Q_{\nu}^{m}(z)$ are independent solutions of Legendre's differential equation

$$(1.17)\qquad \frac{d}{dz}\left((1-z^2)\frac{d}{dz}\right)u+\nu(\nu+1)u-\frac{m^2}{1-z^2}\,u=0.$$

Put

$$(1.18)\qquad \begin{aligned} c_{11}&=-Q_{\nu_2}^{m}(0)=\sqrt{\pi}\,2^{m-1}e^{m\pi i}\\ &\qquad \sin\left(\frac{\nu_2-m}{2}\right)\Gamma\left(\frac{\nu_2+m+1}{2}\right)/\ \Gamma\left(\frac{\nu_2-m}{2}+1\right),\\ c_{12}&=P_{\nu_2}^{m}(0)=\sqrt{\pi}\,2^{m}e^{m\pi i}/\left(\Gamma\left(\frac{\nu_2-m}{2}+1\right)\Gamma\left(\frac{-\nu_2-m+1}{2}\right)\right),\\ c_{21}&=-\frac{d}{dz}Q_{\nu_2}^{m}(0)=-\sqrt{\pi}\,2^{m}e^{m\pi i}\\ &\qquad \cos\left(\frac{\nu_2+m}{2}\right)\Gamma\left(\frac{\nu_2+m}{2}+1\right)/\ \Gamma\left(\frac{\nu_2-m+1}{2}\right),\\ c_{22}&=\frac{d}{dz}P_{\nu_2}^{m}(0)=2^{m+1}e^{m\pi i}\\ &\qquad \sin\left(\frac{\nu_2+m}{2}\pi\right)\Gamma\left(\frac{\nu_2+m}{2}+1\right)/\left(\sqrt{\pi}\ \Gamma\left(\frac{\nu_2-m+1}{2}\right)\right).\end{aligned}$$

Then $c_{12}\neq0$, $c_{22}\neq0$ for $-1<\Re e\ \nu_2<0$. Put

$$(1.19)\qquad R_{\nu_2}^{m}(z)=c_{11}P_{\nu_2}^{m}(z)+c_{12}Q_{\nu_2}^{m}(z),$$

$$S^m_{\nu_2}(z)= c_{21}P^m_{\nu_2}(z)+c_{22}Q^m_{\nu_2}(z).$$

Then we obtain

**Proposition 1.** *Put* $\nu_1=\dfrac{-1+\sqrt{1-4\lambda_1}}{2}$, $\nu_1=\dfrac{-1+\sqrt{1-4\lambda_1}}{2}$, *and assume* $-1<\Re e\ \nu_1<0$, $-1<\Re e\ \nu_2<0$. *Then there exist polynomials* $h_1,h_2$ *of* $y^{-1}$ *of degree* $n-1,n$ *such that*

$$\int_0^\infty \left( \int_1^\infty P^0_{\nu_1}(z_1)R^n_{\nu_2}(z_2)(-2\pi y z_2)^{-n} (z_2{}^2-1)^{n/2}\, e^{-2\pi z_1 z_2 y}\, dz_2 \right) dz_1 - h_1,$$

$$\int_0^\infty \left( \int_1^\infty P^0_{\nu_1}(z_1)S^n_{\nu_2}(z_2)(-2\pi y z_2)^{-n} (z_2{}^2-1)^{n/2}\, e^{-2\pi z_1 z_2 y}\, dz_2 \right) dz_1 - h_2,$$

$$\int_1^\infty \int_1^\infty P^n_{\nu_1}(z_1)P^n_{\nu_2}(z_2)(z_1{}^2-1)^{n/2} (z_2{}^2-1)^{n/2}\, e^{-2\pi z_1 z_2 y}\, dz_1 dz_2,$$

*are linearly independent solutions of the equation* (1.14).

Note that polynomials $h_1,h_2$ can be given explicitly. By the recurrence relation (1.10) we obtain especially

**Theorem 1.** *The functions of* $x,y$

$$C_n(x,y)=(x^2-y^2)\int_1^\infty \int_1^\infty P^n_{\nu_1}(z_1)P^n_{\nu_2}(z_2) J_n\left(2\pi i(z_1{}^2-1)^{1/2}(z_2{}^2-1)^{1/2}x\right) e^{-2\pi z_1 z_2 y} dz_1 dz_2,$$

*are solutions of the equations* (1.8.1), (1.8.2) *where* $J_n$ *denotes the Bessel function of first kind.*

**2.** Here we introduce a certain theta function and show the commutation relations of invariant differential operators and explain their connection with the generalized Whittaker functions. we deal with L-function of biquadratic fields with Grössencharacter at the end.

For $(g_1, g_2) \in SL_2(\mathbf{R}) \times SL_2(\mathbf{R})$, put

$$(2.1) \qquad \rho(g_1, g_2) = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right) \left(\begin{array}{c|c} g_1^{-1} & \\ \hline & g_1^{-1} \end{array}\right) \left(\begin{array}{c|c} a_2E & c_2E \\ \hline b_2E & d_2E \end{array}\right) \left(\begin{array}{cc|cc} 0 & -1 & & \\ 1 & 0 & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right)$$

with $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Then $\rho(g_1, g_2)$ is an element in the orthogonal group for $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$. For an odd squarefree integer N, we define a lattice $\mathscr{L}$ by

$$(2.2) \qquad \mathscr{L} = \left\{ \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \\ m_{41} & m_{42} \end{pmatrix} \;\middle|\; \begin{array}{l} m_{i,j} \in \mathbf{Z} \text{ for } (i,j) \neq (1,2), \\ Nm_{1,2} \in \mathbf{Z} \end{array} \right\}.$$

For $Z = X + iY \in H_2$ and a Dirichlet character $\chi$ modulo N, put

$$(2.3) \qquad \Theta_\chi(Z, (g_1, g_2)) = |Y| \sum_{m = \begin{pmatrix} * & m_{12} \\ * & * \\ * & * \\ * & * \end{pmatrix} \in \mathscr{L}} \chi(Nm_{12}) e^{\pi i N \operatorname{tr}\left(X\left(\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}[\rho(g_1, g_2) m]\right)\right)} e^{-\pi N \operatorname{tr}\left(Y\left(\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}[\rho(g_1, g_2) m]\right)\right)}$$

where S[T] means ${}^tTST$, which is usually called theta function. Put $g_{x+iy}=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix}$ for $x+iy\in H_1$, the upper half plane and put

$$(2.4)\qquad \theta(Z,z_1,z_2)=\Theta_\chi(Z,(g_{z_1},g_{z_2}))$$

then we have

**Theorem 2.** *For* $z_1=x_1+iy_1$, $z_2=x_2+iy_2$,

$$\Delta_1\theta(Z,z_1,z_2)$$
$$=1/8\left(y_1{}^2\left(\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial y_1{}^2}\right)+y_2{}^2\left(\frac{\partial^2}{\partial x_2{}^2}+\frac{\partial^2}{\partial y_2{}^2}\right)\right)\theta(Z,z_1,z_2),$$

$$\Delta_2\theta(Z,z_1,z_2)$$
$$=\left(\frac{1}{256}\left(y_1{}^2\left(\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial y_1{}^2}\right)-y_2{}^2\left(\frac{\partial^2}{\partial x_2{}^2}+\frac{\partial^2}{\partial y_2{}^2}\right)\right)^2\right.$$
$$-\frac{1}{32}\left(y_1{}^2\left(\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial y_1{}^2}\right)+y_2{}^2\left(\frac{\partial^2}{\partial x_2{}^2}+\frac{\partial^2}{\partial y_2{}^2}\right)\right)$$
$$\left.+\frac{3}{64}\right)\theta(Z,z_1,z_2).$$

Note that $\Delta_1,\Delta_2$ defined in 1 are differential operators with respect to Z not to $z_1,z_2$. Let $\varphi_1,\varphi_2$ be Maass wave cusp forms with the character $\chi$ satisfying $y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi_1(z)=\lambda_1\varphi_1(z)$, $y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi_2(z)=\lambda_2\varphi_2(z)$. Put $d_0z=y^{-2}dxdy$ for $z=x+iy$, and define

(2.5) $$F_{\varphi_1,\varphi_2}(Z) = \int_{\Gamma\backslash H_1}\int_{\Gamma\backslash H_1} \theta(Z,z_1,z_2)\ \varphi_1(z_1)\varphi_2(z_2)d_0z_1d_0z_2.$$

with $\Gamma=\Gamma_0(N)$. Then we have

(2.6) $$\Delta_1\ F_{\varphi_1,\varphi_2}(Z) = d_1\ F_{\varphi_1,\varphi_2}(Z),\ \Delta_2\ F_{\varphi_1,\varphi_2}(Z) = d_2\ F_{\varphi_1,\varphi_2}(Z),$$

with $d_1, d_2$ defined by the equalities (1.5) for $\lambda_1, \lambda_2$, and

(2.7) $$F_{\varphi_1,\varphi_2}(\sigma Z) = \chi(d) F_{\varphi_1,\varphi_2}(Z),$$

hold for $\sigma=\left(\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ \hline * & * & a & b \\ * & * & c & d \end{array}\right)\in\Gamma_2$ where

$$\Gamma_2=\left\{\ \sigma=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}\ \middle|\ \begin{array}{l} \sigma\in Sp(2,\mathbf{Q}), \\ a_{21}, a_{31}, a_{32}, a_{41}, a_{42}\in N\mathbf{Z}\ , \\ Na_{13}\in\mathbf{Z},\ \text{other } a_{ij}\in\mathbf{Z} \end{array}\right\}.$$

There exist a lattice $\mathcal{T}$ in $\left\{\begin{pmatrix} a & b \\ b & c \end{pmatrix} \middle| a,b,c\in\mathbf{Q}\right\}$ containing $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$ such that $F_{\varphi_1,\varphi_2}(Z)$ is expanded as follows:

(2.8) $$F_{\varphi_1,\varphi_2}(Z) = \sum_{T\in\mathcal{T}} A(T,Y)e^{2\pi i\,tr(TX)}.$$

Hence $W(Y)=A(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{N} & 0 \\ 0 & 1 \end{pmatrix}^{-1} Y \begin{pmatrix} \sqrt{N} & 0 \\ 0 & 1 \end{pmatrix}^{-1})$ is a generalized Whittaker function. A direct calculation shows that $W(Y)$ is equal to

$\sum_{n\in\mathbb{Z}} b_n B_n(t_1,t_2)e^{ni\theta}$ and $b_0$ is equal to

$$(2.9)\qquad \sum_{j=1}^{h} \chi(a_j)\varphi_1\left(\frac{-1}{\frac{c_jN+\sqrt{-N}}{a_j}}\right)\varphi_2\left(\frac{-1}{\frac{c_jN+\sqrt{-N}}{a_j}}\right)$$

up to a constant multiple, where $\mathfrak{A}_j=a_j\mathbb{Z}+(c_jN+\sqrt{-N})\mathbb{Z}$ runs over the full set of representatives of the ideal class group of $\mathbb{Q}(\sqrt{-N})$ with class number h and $B_n(t_1,t_2)$ is defined by (1.3),(1.7) with $C_n(x,y)$ in Theorem 1.

Note that the corresponding L-function (the spinor L-function) of $F_{\varphi_1,\varphi_2}(Z)$ is the product $L(s,\varphi_1)L(s,\varphi_2)$ of L-functions of $\varphi_1$ and $\varphi_2$. Let $K_i=\mathbb{Q}(\sqrt{d_i})$ be a real quadratic field with discriminant $d_i$ for i=1,2,3. Let $d_3=d_1d_2$ and assume the class number of $K=K_3$ is one. Let $\mathfrak{o}$ be the ring of integers in K and $E_+$ the group of all totally positive units in $\mathfrak{o}$. Put

$$(2.10)\qquad g_1(z,\xi_m)=\sum_{\substack{\mu\in\mathfrak{o}/E_+\\ \mu\neq 0}} \xi_m(\mu)\psi(\mu)y^{1/2} K_{im\kappa}(2\pi|N_{K/\mathbb{Q}}\mu|y)\cos(2\pi xN_{K/\mathbb{Q}}\mu),$$

$$g_2(z,\xi_m)=\sum_{\substack{\mu\in\mathfrak{o}/E_+\\ \mu\neq 0}} \xi_m(\mu)\psi(\mu)^2y^{1/2} K_{im\kappa}(2\pi|N_{K/\mathbb{Q}}\mu|y)\cos(2\pi xN_{K/\mathbb{Q}}\mu)$$

where $\xi_m(\mu)=|\mu/\mu'|^{im\kappa}$, $\kappa=2\pi/\log\varepsilon$, $\psi(\mu)=\left(\frac{d_1}{N_{K/\mathbb{Q}}\mu}\right)$, $\varepsilon$ the fundamental unit, $K_{im\kappa}$ the modified Bessel function. Then $g_1,g_2$ are Maass waveforms and we can take $g_1,g_2$ as $\varphi_1,\varphi_2$ in (2.5). In this case L-function corresponding to $F_{\varphi_1,\varphi_2}$ is

$$\sum_{\mathfrak{a}} \xi(N_{F/k}\mathfrak{a}) N_{F/\mathbb{Q}}\mathfrak{a}^{-s}$$

where $\mathfrak{a}$ runs over all integral ideals in $F=\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ such that $(\mathfrak{a}, d_1 d_2)=1$.

REFERENCES


[1] M. Eichler, Quadratische Formen und orthogonale Groupen, Springrer.

[2] S. Friedberg, Differential operators and theta series, Trans. Amer. Math. Soc., 287 (1985), 569-589.

[3] G. Kaufhold, Dirichletsche Reihe mit Funktionalgleichung in der Theorie der Modulfunktion 2. Grades, Math. Ann. 137 (1959), 454-476.

[4] M. Koecher, Über Thetareihen indefinter quadratischer Formen, Math Nachr. 9 (1953), 51-85.

[5] H. Maass, Die Differentialgleichunen in der Theorie der Siegelschen Modulfunktionen, Math. Ann., Bd. 126 (1953), 44-68.

[6] ————, Über eine neue Art von nichtanalitishen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Functionalgleichungen, Math. Ann. 121 (1949), 141-183.

[7] ————, Dirichletsche Reihen und Modulfunktionen zweiten Grades, Acta Arith., 24 (1973), 225-238.

[8] ————, Über die räumliche Verteilung der Punkte in Gittern mit indefiniter Metrik, Math. Ann. 138 (1959), 287-315.

[9] S. Nakajima, On invariant differential operators on bounded

symmetric domains of type Ⅳ, Proc. Japan Acad., **58**, Ser. A (1982), 235-238.

[10] ————————, Invariant differential operators on SO(2,q)/SO(2)xSO(q) (q≥3), Master these.

[11] G. Shimura, Confluent hypergeometric functions on tube domains, Math. Ann. 265 (1982), 269-302.

[12] H. Yamashita, On Whittaker vectors for generalized Gelfand-Graev representations of semisimple Lie groups, J. Math. Kyoto Univ., 26-2 (1986), 263-298.

[13] ————————, On Whittaker vectors for generalized Gelfand-Graev representations of semisimple Lie groups, Proc. Japan Acad., **61**, Ser. A (1985), 213-216.

[14] ————————, Finite multiplicity theorems for induced representations of semisimple Lie groups and their applications to generalized Gelfand-Graev representations, Proc. Japan Acad., **63**, Ser. A (1987), 153-156.

[15] ————————, Multiplicity one theorems for generalized Gelfand-Graev representations of semisimple Lie groups and Whittaker models for the discrete series, Preprint.

[16] H. Yoshida, Siegel's modular forms and the arithmetic of quadratic forms, Inv. Math. 60 (1980), 193-248.

[17] ——————, On Siegel modular forms obtained from theta series, J. reine angew. Math. 352 (1984), 184-219.