

**Abstract  $L^p$  estimates for the Cauchy problem  
with applications to the Navier-Stokes equations  
in exterior domains**

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**Abstract.**

We strengthen the theory of analytic semigroups in  $\zeta$ -convex Banach spaces and derive global in time a priori  $L^p - L^q$  estimates for solutions of the nonstationary Stokes equations in domains which are not necessarily bounded. We apply these estimates to obtain various new global estimates for weak solutions of the Navier-Stokes equations.

**1. Introduction**

We are concerned with global  $L^p$  estimates of initial-boundary value problems for linear parabolic equations. We are especially interested in the nonstationary Stokes system in a domain  $\Omega$  in  $\mathbb{R}^n$  ( $n \geq 2$ ) with smooth boundary  $\partial\Omega$  (at least  $\partial\Omega \in C^{2+\mu}$ ,  $0 < \mu < 1$ ):

$$(1.1) \quad \frac{\partial u}{\partial t} - \Delta u + \nabla \varphi = f, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0$$

$$(1.2) \quad u(x, 0) = a(x).$$

Here  $u = (u^1(x, t), \dots, u^n(x, t))$  and  $\varphi(x, t)$  represent the unknown velocity and pressure, respectively;  $f$  represents a given external force and  $a$  denotes the initial velocity. For the moment we assume  $a = 0$  to simplify the explanation. When  $n = 3$  and  $\Omega$  is a bounded or an exterior domain, for every  $f \in L^p(\Omega \times (0, T))^n$ ,  $0 < T < \infty$ ,  $1 < p < \infty$ , Solonnikov [41] constructed a unique solution  $(u, \nabla \varphi)$  of (1.1)-(1.2) in  $\Omega \times [0, T)$  satisfying the  $L^p$  estimate

$$(1.3) \quad \int_0^T \left\| \frac{\partial u}{\partial t}(t) \right\|_p^p dt + \int_0^T \left\| \nabla^2 u(t) \right\|_p^p dt + \int_0^T \left\| \nabla \varphi(t) \right\|_p^p dt \leq C \int_0^T \|f(t)\|_p^p dt$$

with  $C = C(T, \Omega, p)$  independent of  $f$ . Here  $\|\cdot\|_p$  denotes the norm in  $L^p(\Omega)$  or  $L^p(\Omega)^m = L^p(\Omega) \times \cdots \times L^p(\Omega)$  with  $m = n$  or  $n^2$  and  $\nabla^2 u = (\partial_i \partial_j u)_{i,j=1,2,\dots,n}$  is the matrix of the second order derivatives of  $u$ . When  $\Omega$  is unbounded, Solonnikov's estimate (1.3) in [41] is not global in time because  $C(T, \Omega, p)$  may tend to infinity as  $T \rightarrow \infty$ . His approach is based on methods in the theory of partial differential equations, in particular potentials, and it seems difficult to extend his method to get a global estimate i.e. the estimate (1.3) with  $C$  independent of  $T$ .

This paper strengthens such  $L^p$  estimates for parabolic equations in two directions. First, our estimate is global in time. Secondly, the integral norms we use have different exponents in space and time. For example let us consider the Stokes system in an exterior domain  $\Omega$  in  $\mathbb{R}^n$  with  $n \geq 3$ . We shall prove that for every  $f \in L^p(0, t; L^q(\Omega)^n)$ ,  $0 < T \leq \infty$ ,  $1 < q < \infty$ , there is a unique solution  $(u, \varphi)$  of (1.1)-(1.2) so that

$$(1.4) \quad \int_0^T \left\| \frac{\partial u}{\partial t}(t) \right\|_q^p dt + \int_0^T \left\| \nabla^2 u(t) \right\|_q^p dt + \int_0^T \left\| \nabla \varphi(t) \right\|_q^p dt \leq C \int_0^T \|f(t)\|_q^p dt$$

with  $C = C(\Omega, p, q)$  independent of  $T$  and  $f$  provided  $1 < q < n/2$ ;  $a = 0$  is assumed for simplicity. Here  $L^p(0, T; X)$  denotes the space of  $L^p$  functions in  $(0, T)$  with values in a Banach space  $X$ . Since  $C$  does not depend on  $T$ , we may include the case  $T = \infty$ ; this gives new global properties of the solution  $u$ .

To derive global  $L^q - L^p$  estimates such as (1.4) we extend an abstract parabolic semigroup theory recently developed by Dore and Venni [13]. Let us first review their theory. We consider an ordinary differential equation for functions with values in a Banach space  $X$ :

$$(1.5) \quad u' + Au = f, \quad u(0) = 0, \quad (u' = du/dt),$$

where  $A$  is a densely defined closed linear operator in  $X$ . The operator  $A$  is assumed to be nonnegative, i.e., no negative real number is in the spectrum of  $A$  and the operator norm of  $t(t+A)^{-1}$  is bounded in  $t > 0$ . We also assume that the pure imaginary powers  $A^{is}$  are bounded linear operators and their operator norm is estimated by

$$(1.6) \quad \|A^{is}\| \leq K e^{\theta|s|}, \quad s \in \mathbb{R}$$

with some  $K \geq 1$  and  $\theta$  satisfying

$$(1.7) \quad 0 \leq \theta < \pi/2$$

independent of  $s$ . In [13] Dore and Venni proved that when  $A$  has a bounded inverse, the equation (1.5) has a unique solution for given  $f \in L^p(0, T; X)$ ,  $0 < T < \infty$ ,  $1 < p < \infty$  such that

$$(1.8) \quad \int_0^T \|u'(t)\|_X^p dt + \int_0^T \|Au(t)\|_X^p dt \leq C \int_0^T \|f(t)\|_X^p dt$$

with  $C = C(T, p, X)$  provided that (1.6) with (1.7) holds and that  $X$  is  $\zeta$ -convex. In this paper we extend their theory to the case where  $A$  may not have a bounded inverse. Moreover we show that (1.8) holds with  $C$  independent of  $T$ , so we obtain a global estimate if  $A$  has a dense range.

As in [13], the estimate (1.8) is reduced to properties of the inverse  $(A + B)^{-1}$  when both  $A$  and  $B$  are nonnegative and satisfy (1.6) with  $\theta_A$  and  $\theta_B$ , respectively. Assuming that  $A$  and  $B$  are *resolvent commuting*, i.e.,

$$(1.9) \quad (t + A)^{-1}(t + B)^{-1} = (t + B)^{-1}(t + A)^{-1}, \quad \text{for all } t > 0,$$

we prove a fundamental result (extending that of [13]). It reads:

$$(1.10) \quad \begin{aligned} &\text{if } \theta_A + \theta_B < \pi, \quad \text{then } (A + B)^{-1} \text{ is bounded} \\ &\text{from } X \text{ to } \hat{D}(A + B) \text{ provided that} \\ &X \text{ is } \zeta\text{-convex and the ranges of } A \text{ and } B \\ &\text{are dense in } X. \end{aligned}$$

Here  $\hat{D}(A + B)$  is the completion of the intersection of domains of  $A$  and  $B$  under the norm  $\|Au\| + \|Bu\|$ . The idea of the proof is basically similar to that in [13]. However, since  $A^z$  is in general not a bounded operator even if  $\text{Re } z < 0$ , we should be careful to understand the commutativity of  $A^z$  and  $B^w$  from (1.9) as well as a choice of  $g$  such that  $A^z B^w g$  is well-defined. In this paper we give a whole proof of (1.10). Recently, Prüss and the second author [33] found another proof of essentially the same result as (1.10).

We now apply our estimate (1.8) with  $C = C(p, X)$  for the Stokes system (1.1)-(1.2). As is well known, the system can be transformed into (1.5) by taking  $A$  as the Stokes operator with Dirichlet condition in

$$X = L^q_\sigma(\Omega) = \{u \in L^q(\Omega)^n; \operatorname{div} u = 0, \quad u \cdot \nu|_{\partial\Omega} = 0\},$$

where  $1 < q < \infty$  and  $\nu$  is the outer normal vector in  $\partial\Omega$ . For the Stokes operator the estimate of imaginary powers (1.6) is known for every  $\theta > 0$ . This estimate was first proved by [20] when  $\Omega$  is bounded and by [23] when  $\Omega$  is an exterior domain in  $\mathbb{R}^n$  with  $n \geq 3$ . We show in Appendix that the same estimate holds when  $\Omega$  is a halfspace using results in [3]. Since  $L^q(\Omega)$  and also  $L^q_\sigma(\Omega)$  are typical examples of  $\zeta$ -convex spaces (cf. [13]), our abstract theory is applicable. We obtain (1.8) with  $X = L^q_\sigma(\Omega)$  and the Stokes operator. The estimate (1.4) with  $C = C(\Omega, p, q)$  easily follows if we apply the a priori estimate  $\|\nabla^2 u\|_q \leq C\|Au\|_q$  for  $1 < q < n/2$  due to Solonnikov [41] (for  $n = 3$  and [23] for  $n \geq 3$ ) when  $\Omega$  is exterior; the restriction  $q < n/2$  is unnecessary when  $\Omega$  is bounded or a halfspace.

The estimate (1.4) is important in studying regularity and large time behavior of weak (or strong) solutions of the nonstationary Navier-Stokes system in an exterior domain

$$(1.11) \quad \begin{aligned} \frac{\partial v}{\partial t} - \Delta v + (v, \nabla)v + \nabla\psi &= f, \quad \operatorname{div} v = 0, \quad v|_{\partial\Omega} = 0, \\ v(x, 0) &= a(x), \quad (v, \nabla) := \sum_{i=1}^n v^i \partial_i, \quad \partial_i = \partial/\partial x_i \end{aligned}$$

and recently this problem is extensively studied [4, 5, 18, 24, 27, 28, 30, 31, 38, 39, 40]. Although a global weak solution under suitable assumptions on  $a$  and  $f$  is known to exist, its regularity is not known unless  $n = 2$ . As an application of (1.4) we derive various global a priori estimates both for  $v$  and  $\psi$ . For example, when  $\Omega$  is an exterior domain in  $\mathbb{R}^n$  with  $n \geq 3$ , we have

$$(1.12) \quad \int_0^\infty \|\nabla^2 v(t)\|_q^s dt + \int_0^\infty \|\nabla\psi(t)\|_q^s dt < \infty$$

for  $n+1 = n/q + 2/s$ ,  $1 < q, s < \infty$ , provided that  $v$  solves (1.11) in the weak sense and that  $v \in L^2(0, \infty; L^2_\sigma(\Omega))$ ,  $\nabla v \in L^2(\Omega \times (0, \infty))^{n^2}$  and  $f \in L^s(0, \infty; L^q_\sigma(\Omega))$  (and  $a = 0$  for

simplicity). The estimate (1.12) implies that one can choose the pressure  $\psi$  so that

$$(1.13) \quad \int_0^\infty \|\psi(t)\|_r^s dt < \infty, \quad n/r + 2/s = n.$$

We note that this pressure estimate for  $n = 3$ ,  $r = s = 5/3$  simplifies the partial regularity theory of suitable weak solutions in [9]. The global result (1.13) is new while  $\psi \in L^s(0, T; L^r(\Omega))$  for finite  $T$  is known by [39]. From (1.12) we also deduce a global decay result of  $v$ :

$$(1.14) \quad \int_0^\infty \|v(t)\|_k^\rho dt < \infty, \quad n/k + 2/\rho = n - 1, \quad \rho \geq s, \quad k \geq q.$$

Although there are many results on large time behavior of weak solutions (see e.g. [4]) our result (1.14) is not contained in the literature because (i) our estimate holds for weak solutions which need not satisfy energy inequalities and (ii) we estimate the decay as  $t \rightarrow \infty$  by an integral norm while the algebraic decay order of  $\|v(t)\|_k$  as  $t \rightarrow \infty$  is studied in the literature.

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