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HEAT CONVECTION EQATIONS IN NONCYLINDORICAL DOMAINS

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§1 Introduction.

Let $Q(t)$ be a bounded domain in $\mathbb{R}^N$ ($N=2$ or 3) with smooth boundary $\Gamma(t)$ for each $t\in[0,T]$, T be any positive number.

Consider the following Oberbeck-Boussinesq equations in the noncylindrical domain $Q = \bigcup_{0\le t\le T} Q(t)\times\{t\}$ with lateral boundary $\Gamma = \bigcup_{0\le t\le T} \Gamma(t)\times\{t\}$:

$$
(1.1)\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u = -\dfrac{\nabla p}{\rho} + \{1-\eta(\theta-d)\}g & (x,t)\in Q,\\ \operatorname{div} u = 0 & (x,t)\in Q,\\ \theta_t - \kappa\Delta\theta + (u\cdot\nabla)\theta = 0 & (x,t)\in Q,\end{cases}
$$

$$(1.2)\quad u(x,t) = \alpha(x,t),\quad \theta(x,t) = \beta(x,t) \qquad (x,t)\in\Gamma,$$

$$(1.3)\quad u(x,0) = u_0(x),\quad \theta(x,0) = \theta_0(x) \qquad x\in Q(0),$$

where $(u\cdot\nabla) = \sum_{j=1}^{N} u^j \dfrac{\partial}{\partial x_j}$. Unknown functions $u=(u^1,u^2,\cdots,u^N)$, p and θ are the solenoidal velocity, pressure and temperature of the fluid which occupies Q respectively; α, β, u_0, θ_0 are given data and g is the body force field (say gravity); constants ν, ρ, κ, η, d represent kinematic viscosity, density ,thermal conductivity, volume expansion coefficient and some datum point of the temperature of the fluid respectively,(see Joseph [6]). In what follows, the special case $\nu = \kappa = 1$ will be treated for the sake of simplicity.

The purpose of the present paper is to investigate the existence of local and global solutions and their regularity. This kind of problem has been studied by several authors. As for the case where Q is a cylindrical domain, $Q_o\times[0,T]$, Kirchgässner and Kielhöfer [7] and Chidaglia [4] constructed local and global strong solutions in some Sobolev spaes. Recently, Morimoto [8] discussed the existence of weak solutions of the equation with the boundary condition for θ replaced by a discontinuous Neumann-Dirichlet condition, and Hishida [5] the existence of strong solutions in the $L^p x L^q$ space. The problem in noncylindrical domains was studied by Ōeda [9], where the existence of weak and strong solutions is discussed.

As far as the existence of strong solutions in L^2-framework is concerned, our results ameliorate those above even for the case where Q is cylindrical. Our method of proofs relies on the theory of perturbation for time-dependent subdifferential operators based on nonlinear interpolation theory, developed in Ôtani [10], which is different from those of papers cited above.

Our main results are stated in the next section and their proofs are given in § 3.

§2 Main Results.

In order to formulate our results, we here fix some notations. We denote by $H^s(\Omega)$ the sobolev space of order s in the Lebesgue space $H(\Omega) = L^2(\Omega)$ with norm $|\cdot|$, and set

$\mathbb{C}_\sigma^\infty(\Omega)=\{u=(u^1,\cdots,u^N) \mid u^j \in C_0^\infty(\Omega), (j=1,\cdots,N), \mathrm{div}\, u = 0\}$,

$\mathbb{H}(\Omega)=(H(\Omega))^N$ with norm $\|\cdot\|$,

$\mathbb{H}^s(\Omega)=(H^s(\Omega))^N$ with norm $\|\cdot\|_{H^s}$,

$\mathbb{H}_\sigma^1(\Omega)=\mathbb{H}^1(\Omega)\cap\mathbb{H}_\sigma(\Omega)$,

$\mathbb{H}_\sigma(\Omega)$: the completion of $\mathbb{C}_\sigma^\infty(\Omega)$ under the $\mathbb{H}(\Omega)$-norm,

P_Ω: the orthogonal projection from $\mathbb{H}(\Omega)$ onto $\mathbb{H}_\sigma(\Omega)$,

$A(\Omega)= -P_\Omega\Delta$:Stokes operator with domain $D(A(\Omega))= \mathbb{H}^2(\Omega)\cap\mathbb{H}_\sigma^1(\Omega)$,

$A^\mu(\Omega)$: the fractional power of $A(\Omega)$ of order μ, whose domain is characterized by Fujita and Morimoto [2] and Fujiwara [3]. We also use the notations:

$\|u\|_p = \|u\|_{L^p}$, $|\theta|_p = |\theta|_{L^p}$,

$$\|u\|_{2,T}^2 = \max_{0\le t\le T} \int_0^T \|u(t)\|^2 dt, \quad |\theta|_{2,T}^2 = \max_{0\le t\le T} \int_0^T |\theta(t)|^2 dt,$$

$$\|u\|_{\infty,T} = \sup_{0\le t\le T} \|u(t)\|, \quad |\theta|_{\infty,T} = \sup_{0\le t\le T} |\theta(t)|,$$

$$\|u\|_{M,T}^2 = \sup_{1\le t\le T} \int_{t-1}^t \|u(s)\|^2 ds, \quad |\theta|_{M,T}^2 = \sup_{1\le t\le T} \int_{t-1}^t |\theta(s)|^2 ds.$$

We assume that Q is smooth and α and β can be extended to Q in the following sense:

(A.Q) There exists a level preserving C^3-diffeomorphism $\mathcal{G}$ from Q onto $Q_o\times[0,T]$ for some bounded domain Q_o in $\mathbb{R}^N$.

(A.α) There exists a vector function $\bar{u}$ in $C^1(Q)$ such that $\bar{u}\in L^\infty(0,T;\mathbb{H}(Q(t)))\cap L^6(0,T;\mathbb{H}^1(Q(t)))$; $\bar{u}_t,\Delta\bar{u}\in L^2(0,T;\mathbb{H}(Q(t)))$, $\mathrm{div}\,\bar{u}=0$ in Q and $\bar{u}=\alpha$ on Γ.

(A.β) There exists a function $\bar{\theta}$ in $C^1(Q)$ such that $\bar{\theta}\in L^6(0,T;H^1(Q(t)))$, $\bar{\theta}_t,\nabla\bar{\theta}\in H(Q)$ and $\bar{\theta}=\beta$ on Γ.

(A.g) g has the potential $G\in L^\infty(0,T;W^{1,\infty}(Q(t)))$ i.e., $g=\nabla G$. (When g is the gravity, this condition is always satisfied.)

Let B be a bounded domain in $\mathbb{R}^N$ such that the closure of Q is contained in $B\times[0,T]$. We mean by $C(I;X(Q(t)))$ the set of all functions v on Q such that $v(\cdot,t)$ belongs to $X(Q(t))$ for all $t\in I$ and the zero extension $\hat{v}$ of v to $B\times[0,T]$ is an $X(B)$-valued continuous function on I, where I is an interval in $[0,T]$ and $X(\Omega)$ is a function space defined on Ω such as $\mathbb{H}(\Omega)$, $\mathbb{H}_\sigma(\Omega)$,etc. Now our main results are stated as follows:

Theorem I (Global existence for N=2) Let (A.α), (A.β) and (A.g) be satisfied, and let $u_o - \bar{u}(\cdot,0) \in D(A^{\mu}(Q(0)))$ with $\mu \in (0,\frac{1}{2})$ and $\theta_o - \bar{\theta}(\cdot,0) \in H(Q(0))$. Then (1.1)-(1.3) has a unique solution (u,θ) satisfying

$$(\#.u.\mu) \quad \begin{cases} u - \bar{u} \in C([0,T];\mathbb{H}_{\sigma}(Q(t))) \cap C((0,T];\mathbb{H}^1_{\sigma}(Q(t))), \\ t^{\frac{1}{2}-\mu}\, u_t,\ t^{\frac{1}{2}-\mu}\, \Delta u \in L^2(0,T;\mathbb{H}(Q(t))). \end{cases}$$

$$(\#.\theta) \quad \begin{cases} \theta - \bar{\theta} \in C([0,T];H(Q(t))), \\ t^{\frac{1}{2}}\, \theta_t,\ t^{\frac{1}{2}}\, \Delta\theta \in H(Q). \end{cases}$$

Theorem II (Local existence for N=3) Let (A.Q),(A.α) and (A.β) be satisfied, and let $u_o - \bar{u}(\cdot,0) \in D(A^{\frac{1}{4}}(Q(0)))$ and $\theta_o - \bar{\theta}(\cdot,0) \in H(Q(0))$. Then there exists a positive number T_o depending on $\|A^{\frac{1}{4}}(Q(0))(u_o - \bar{u}(\cdot,0))\|$ and $|\theta_o - \bar{\theta}(\cdot,0)|$ (and also on $\bar{u}$ and $\bar{\theta}$) such that (1.1)-(1.3) has a unique solution (u,θ) on $[0,T_o]$ satisfying (#.u.1/4) and (#.θ) with T replaced by T_o.

Theorem III (Global existence for N=3) Let the same assumptions in Theorem II be satisfied. Then there exists a (sufficiently small) positive number r_o depending on $|\theta_o|, |\bar{\theta}_t|_{M,T}, |\Delta\bar{\theta}|_{M,T}$ and $||\nabla\bar{\theta}|^2|_{M,T}$ such that if $\|A^{\frac{1}{4}}(Q(0))(u_o - \bar{u}(\cdot,0)\|, \|\bar{u}_t\|_{M,T}, \|\Delta\bar{u}\|_{M,T}, \|\nabla\bar{u}\|_{M,T}, \|\bar{u}\|_{M,T}, \|g\|_{\infty,T} < r_o$, then (1.1)-(1.3) has a unique solution (u,θ) on [0,T] satisfying (#.u.1/4) and (#.θ).

§.3. Proofs of Theorems

3.1 Reduction to Abstract Equations.

In this subsection we are going to show that (1.1)-(1.3) can be reduced to abstract equations in $\mathbb{H}_\sigma(B)$ and H(B) as in [10]and [11]. To this end, we put

$$\varphi_1^t(u) = \varphi_1(u) + I_1^t(u) \qquad u \in \mathbb{H}_\sigma(B),$$

$$\varphi_1(u) = \begin{cases} \frac{1}{2}\int_B |\nabla u|^2 dx & u \in \mathbb{H}_\sigma^1(B), \\ +\infty & u \in \mathbb{H}_\sigma(B) \setminus \mathbb{H}_\sigma^1(B), \end{cases}$$

$$K_1(t) = \{u \in \mathbb{H}_\sigma(B) \mid u = 0 \quad \text{a.e. } x \in B \setminus Q(t)\}.$$

$$I_1^t(u) = \begin{cases} 0 & u \in K_1(t), \\ +\infty & u \in \mathbb{H}_\sigma(B) \setminus K_1(t), \end{cases}$$

$$\varphi_2^t(\theta) = \varphi_2(\theta) + I_2^t(\theta) \qquad \theta \in H(B),$$

$$\varphi_2(\theta) = \begin{cases} \frac{1}{2}\int_B |\nabla \theta|^2 dx & \theta \in H_0^1(B), \\ +\infty & \theta \in H(B) \setminus H_0^1(B), \end{cases}$$

$$K_2(t) = \{\theta \in H(B) \mid \theta = 0 \quad \text{a.e. } x \in B \setminus Q(t)\}.$$

$$I_2^t(\theta) = \begin{cases} 0 & \theta \in K_2(t), \\ +\infty & \theta \in H(B) \setminus K_2(t). \end{cases}$$

Then φ_1^t and φ_2^t are lower semicontinuous convex functions and their subdifferentials are characterized as follows:

$\partial\varphi_1^t(u) = \{f \in \mathbb{H}_\sigma(B) \mid P_{Q(t)} f|_{Q(t)} = A(Q(t))u|_{Q(t)}\}$ with domain

$D(\partial\varphi_1^t)=\{u\in\mathbb{H}_\sigma(B) \mid u|_{Q(t)}\in\mathbb{H}^2(Q(t))\cap\mathbb{H}_\sigma^1(Q(t)),\ u|_{B\setminus Q(t)}=0\}$,

hence $\|\partial\mathring{\varphi}_1^t(u)\|=\|A(Q(t))u|_{Q(t)}\|$, where $\partial\mathring{\varphi}_1^t$ denotes the minimal section of $\partial\varphi_1^t$.

$\partial\varphi_2^t(\theta)=\{h\in H(B) \mid h|_{Q(t)}=-\Delta\theta|_{Q(t)}\}$ with domain

$D(\partial\varphi_2^t)=\{\theta\in H(B) \mid \theta|_{Q(t)}\in H^2(Q(t))\cap H_o^1(Q(t)),\ \theta|_{B\setminus Q(t)}=0\}$,

hence $\|\partial\mathring{\varphi}_2^t(\theta)\|=\|-\Delta\theta|_{Q(t)}\|$.

Furthermore we put

$A_1^t=\partial\varphi_1^t$, $A_2^t=\partial\varphi_2^t$,

$B_1^t(u)=P_B\{(u\cdot\nabla)u+(\bar{u}\cdot\nabla)u+(u\cdot\nabla)\bar{u}\}$,

$B_2^t(u,\theta)=(u\cdot\nabla)\theta+(\bar{u}\cdot\nabla)$

$F_1(t)=P_B\{-\bar{u}_t+\Delta\bar{u}-(\bar{u}\cdot\nabla)\bar{u}-\eta\bar{\theta}g\}$,

$F_2(u,t)=F_2(t)-(u\cdot\nabla)\bar{\theta}$, $F_2(t)=-\bar{\theta}_t+\Delta\bar{\theta}-(\bar{u}\cdot\nabla)\bar{\theta}$,

and consider the following abstract equations in $\mathbb{H}_\sigma(B)$ and $H(B)$;

$$(3.1)\quad \begin{cases} -\hat{u}_t-B_1^t(\hat{u})+F_1(t)-P_B\eta\hat{\theta}g\in A_1^t\hat{u}, \\ \hat{u}(0)=\hat{u}_o=u_o-\bar{u}(\cdot,0), \end{cases}$$

$$(3.2)\quad \begin{cases} -\hat{\theta}_t-B_1^t(\hat{u},\hat{\theta})+F_2(\hat{u},t)\in A_2^t\hat{\theta}, \\ \hat{\theta}(0)=\hat{\theta}_o=\bar{\theta}_o-\theta(\cdot,0). \end{cases}$$

Here it is understood that all the functions defined only on $Q(t)$ (such as $\hat{u}_o,\hat{\theta}_o,\bar{u},\bar{u}_t$,etc) are extended to B by zero. If (3.1)-(3.2) has a solution $(\hat{u},\hat{\theta})$ satisfying ;

$$(3.3)\quad \begin{cases} \hat{u} \in C([0,T];\mathbb{H}_\sigma(B)) \cap C((0,T];\mathbb{H}^1_\sigma(B)), \\ t^{\frac{1}{2}-\mu}\, \hat{u}_t,\ t^{\frac{1}{2}-\mu}\, B^t_1(\hat{u}) \in L^2(0,T;\mathbb{H}_\sigma(B)), \\ \hat{u}(t) \in D(\partial\varphi^t_1) \quad \text{a.e. } t\in[0,T], \end{cases}$$

$$(3.4)\quad \begin{cases} \hat{\theta} \in C([0,T];H(B)) \cap C((0,T];H^1_0(B)), \\ t^{\frac{1}{2}}\, \hat{\theta}_t,\ t^{\frac{1}{2}}\, B^t_2(\hat{u},\hat{\theta}) \in L^2(0,T;H(B)), \\ \hat{\theta}(t) \in D(\partial\varphi^t_2) \quad \text{a.e. } t\in[0,T], \end{cases}$$

Then it is easy to see that $(u,\theta)=(\hat{u}|_{Q(t)}+\bar{u},\hat{\theta}|_{Q(t)}+\bar{\theta})$ gives a solution of (1.1)-(1.3) satisfying $(\#,u,\mu)$ and $(\#,\theta)$. So, in the following, we are going to construct solutions of (3.1)-(3.2) satisfying (3.3)-(3.4).

3.2 Local Existence.

In what follows, we denote $\hat{u},\hat{\theta},\hat{u}_o,\hat{\theta}_o$ by u,θ,u_o,θ_o again. For each $R>0$ and $S\in(0,T]$ set

$$K_{R,S}=\{h\in C([0,S];H(B));\ |h|_{\infty,S} \le R\ \}.$$

Then, for sufficiently small S, we can show the following facts which assure the existence of local solution (u,θ) of (3.1) -(3.2) satisfying (3.3)-(3.4).

(Fact.I) For any $\theta\in K_{R,S}$, there exists a unique solution $u=u_\theta$ of (3.1) with $\hat{\theta}$ replaced by θ satisfying (3.3).

(Fact.II) There exists a unique solution $\tilde{\theta}=\tilde{\theta}_{u_\theta}$ of (3.2) with $\hat{u}$ replaced u_θ satisfying (3.4). So we can define the operator $\mathcal{F}$ by

$$\mathcal{F}:\ \theta \longrightarrow u_\theta \longrightarrow \tilde{\theta}_{u_o}.$$

(Fact.Ⅲ) $\mathcal{F}$ is a contraction from $K_{R,S}$ into $K_{R,S}$.

Proof of Fact.Ⅰ. By (A.α) and (A.β), it is clear that $F_1(t)- P_B\eta\theta g \in L^2(0,S;\mathbb{H}_\sigma(B))$. Let us note

$$(3.5)\quad \|(v\cdot\nabla)w\|^2 \le \begin{cases} C\ \|v\|_4^2\ \|\nabla w\|_4^2 \le C\ \|v\|\|v\|_{H^1}\|\nabla w\|\|\nabla w\|_{H^1} & N=2, \\ C\ \|v\|_6^2\|\nabla w\|_6\|\nabla w\| \le C\ \|v\|_{H^1}^2\|w\|_{H^1}\|\nabla w\|_{H^1} & N=3, \end{cases}$$

where we used the inequality $\|u\|_4^2 \le C\ \|u\|\|u\|_{H^1}$ for $N=2$. Therefore, for any $\varepsilon>0$, there exists a constant C_ε such that

$$(3.6)\ \|B_1^t(u)\|^2 \le \begin{cases} C(\|u\|+1)\{\varepsilon\|\mathring{A}_1^t u\|^2 + C_\varepsilon(\|\nabla u\|^4 + \|\bar{u}\|_{H^1}^4 + \|\bar{u}\|_{H^2}^2 + \|\bar{u}\|^4\|\bar{u}\|_{H^1}^4)\} & N=2 \\ \varepsilon\|\mathring{A}_1^t(u)\|^2 + C_\varepsilon(\|\nabla u\|^6 + \|\bar{u}\|_{H^1}^6 + \|\bar{u}\|_{H^2}^2) & N=3 \end{cases}$$

where $\mathring{A}_1^t$ denotes the minimal section of A_1^t. Then the same argument as in the proof of Theorem 5.1 in [10] assures that there exists a (sufficiently small) number S depending on $\|A^{\frac{1}{4}}(Q(0))u_o\|$ such that (3.1) has a unique solution $u = u_\theta$ satisfying (3.3) with T replaced by S. Furthermore u enjoys the following more minute estimates:

$$(3.7)\quad \begin{cases} t^{\frac{1}{2}-\mu}\|\mathring{A}_1^t u(t)\|\in L^2(0,S),\ t^{\frac{1}{2}-\mu}\|\nabla u(t)\|\in L_*^q(0,S)\ \ \forall q\in[2,\infty] & \text{if } N=2, \\ t^{\frac{1}{4}}\|\mathring{A}_1^t u(t)\|\in L^2(0,S),\ t^{\frac{1}{4}}\|\nabla u(t)\|\in L_*^q(0,S)\ \ \forall q\in[2,\infty] & \text{if } N=3, \end{cases}$$

where $L_*^\infty = L^\infty$ and $L_*^q(0,S)= L^q(0,S;t^{-1}dt)$ for $q\in[2,\infty)$.

Proof of Fact Ⅱ. By much the same verification as for (3.5), we get

$$(3.8)\quad |(v\cdot\nabla)\eta|^2 \le \begin{cases} C\ \|v\|\|v\|_{H^1}|\nabla\eta||\nabla\eta|_{H^1} & \text{if } N=2, \\ C\ \|v\|_{H^1}^2|\nabla\eta||\nabla\eta|_{H^1} & \text{if } N=3. \end{cases}$$

Then, by virtue of (A.α),(A.β) and (3.7), we deduce

(3.9) $\quad t^{\delta/2} F_2(u,t) \in L^2(Q) \quad$ for $\forall\delta\in[\frac{1}{6},1]$,

(3.10) $\quad |B_2^t(u,\theta)|^2 \le \frac{1}{4}|A_2^t\theta|^2 + C\,\varphi_2^t(\theta)\cdot a(t)$,

$$\text{with}\quad a(t)=\begin{Bmatrix}\|u(t)\|^2\|\nabla u(t)\|^2+\|\bar{u}(t)\|^4_{H^1} & (N=2)\\ \|\nabla u(t)\|^4+\|\bar{u}(t)\|^4_{H^1} & (N=3)\end{Bmatrix} \in L^1(0,S).$$

Since $(B_2^t(u,\theta),\theta)_{L^2}= 0$, the same argument as in the proof of Theorem Ⅳ in [10] assures that for any $\theta_o\in D(\varphi_2^o)$, there exist two strong solutions $\eta,\theta^\varepsilon\in \mathscr{L}:=\{\theta\in W^{1,2}(0,S;H(B));\ \Delta\theta|_{Q(t)}\in L^2(0,S;H(Q(t)))\}$ of the following equations :

(3.11) $\quad \eta_t + A_2^t\eta + B_2^t(u,\eta) \ni 0,\ \eta(0)= \theta_o$,

(3.12)$_\varepsilon$ $\quad \theta^\varepsilon_t + A_2^t\theta^\varepsilon+ B_2^t(u,\theta^\varepsilon) \ni \delta_\varepsilon(t)F_2^t(u,t),\ \theta^\varepsilon(0)= \theta_o$,

where $\delta_\varepsilon(t)= 0$ for $0\le t\le\varepsilon$ and $\delta_\varepsilon(t)= 1$ for $t>\varepsilon$.

Then $w^\varepsilon= \eta - \theta^\varepsilon$ satisfies

(3.13) $\quad w^\varepsilon_t + A_2^tw^\varepsilon + B_2^t(u,w^\varepsilon) \ni \delta_\varepsilon(t)F_2^t(u,t),\ w^\varepsilon(0)= 0$,

Multiplying (3.13) by w^ε and $g^\varepsilon= - w^\varepsilon_t+B_2^t(u,w^\varepsilon)+\delta_\varepsilon(t)F_2^t(u,t)\in A_2^tw^\varepsilon$, we obtain

(3.14) $\quad \max\limits_{0\le s\le t} |w^\varepsilon(t)|^2+ \int_0^t|\nabla w^\varepsilon(t)|^2ds \le (\int_0^t|F_2(u,s)|ds)^2 \quad \forall t\in[0,S]$,

(3.15) $\quad \frac{d}{dt}\,\varphi_2^t(w^\varepsilon(t))+|g^\varepsilon(t)|^2$

$$\le m\{|g^\varepsilon(t)|\varphi_2^t(w^\varepsilon(t))^{\frac{1}{2}}+ \varphi_2^t(w^\varepsilon(t))\}+(|B_2^t(u,w^\varepsilon)|+|F_2(u,t)|)|g^\varepsilon(t)|,$$

where we used the fact that there exists a constant m such that

(3.16) $\left|\frac{d}{dt}\varphi_2^t(\theta(t)) - (g_2, \frac{d}{dt}\theta(t))_{H(B)}\right| \leq m\{|g_2|\varphi_2^t(\theta(t))^{\frac{1}{2}} + \varphi_2^t(\theta(t))\}$,

for all $\theta \in \mathscr{L}$, $g_2 \in \partial\varphi_2^t(\theta(t))$ and a.e. $t \in [0,S]$, (for a proof see Lemma 3.6 of [11]). Then ,by (3.10) and (3.15), we get for $\gamma \in [0,1]$

(3.17) $$\frac{d}{dt}t^\gamma\varphi_2^t(w^\varepsilon) + \frac{1}{4}t^\gamma|g^\varepsilon|^2 \leq C(a(t)+1)t^\gamma\varphi_2^t(w^\varepsilon) + t^\gamma|F_2(u,t)|^2 + \gamma t^{\gamma-1}\varphi_2^t(w^\varepsilon).$$

Hence it follows from (3.14),(3.17) with $\gamma=1$ and Gronwall's inequality that

$$t\varphi_2^t(w^\varepsilon(t)) \leq H(t) := C\{\int_0^t s|F_2(u,s)|^2 ds + (\int_0^t |F_2(u,s)|ds)^2\}.$$

By using (3.9) and Hardy's inequality, we can show that

$t^{-1-\mu}H(t) \in L^1(0,S)$ for $\mu \in [0,\frac{5}{6}]$. Consequently we have

(3.18) $$\int_0^S t^{-\mu}\varphi_2^t(w^\varepsilon(t))dt \leq C \text{ (independent of } \varepsilon) \qquad \forall\mu \in [0,\tfrac{5}{6}].$$

Thus (3.17) with $\gamma=1-\mu$ and (3.18) give

(3.19) $$\sup_{0\leq t\leq S} t^\gamma\varphi_2^t(w^\varepsilon(t)) + \int_0^S t^\gamma|g^\varepsilon(t)|^2 dt \leq C \qquad \forall\gamma \in [\tfrac{1}{6},1].$$

Since $\eta \in \mathscr{L}$, (3.14) and (3.19) imply

(3.20) $$\sup_{0\leq t\leq S}\{|\theta^\varepsilon(t)| + t^\gamma\varphi_2^t(\theta^\varepsilon(t))\} + \int_0^S t^\gamma|g_2^\varepsilon(t)|^2 dt \leq C \qquad \forall\gamma \in [\tfrac{1}{6},1],$$

where $g_2^\varepsilon = -\theta_t^\varepsilon + B_2^t(u,\theta^\varepsilon) + \delta_\varepsilon(t)F_2^t(u,t) \in A_2^t\theta^\varepsilon$.

Furthermore, in view of (3.19),(3.9) and (3.10), we obtain

(3.21) $\int_0^S t^\gamma |\theta_t^\varepsilon(t)|^2 dt \le C \quad \forall\gamma\in[\frac{1}{6},1].$

Then it easily follows from (3.20) and (3.21) that $\{\theta^\varepsilon(t)\}_{\varepsilon>0}$ forms a precompact set in $H(B)$ for each $t\in[0,S]$ and is equicontinuous in $C([0,S];H(B))$. Hence, by Ascoli's theorem, we can choose a sequence ε_n which tends to 0 as $n\to\infty$ such that θ^{ε_n} converges to θ in $C([0,S];H(B))$ and the standard argument assures that θ is a solution of $(3.12)_o$, i.e. $(3.12)_\varepsilon$ with δ_ε replaced by 1.

For any $\theta_o\in H(B)$, take $\theta_o^n\in D(\varphi_2^o)$ such that $\theta_o^n\to\theta_o$ in $H(B)$ as $n\to\infty$ and let θ^n be the solution of $(3.12)_o$ with $\theta^n(0)=\theta_o^n$. Then $w=\theta^n-\theta^m$ satisfies

(3.22) $\quad w_t + A_2^t w + B_2^t(u,w) \ni 0,\ w(0)=\theta_o^n-\theta_o^m.$

By the same verifications for (3.14) and (3.19) with $\gamma=1$, we get

(3.23) $\quad \sup_{0\le t\le S}|w(t)|^2 + \int_0^S|\nabla w(t)|^2 dt \le |\theta_o^n-\theta_o^m|^2,$

(3.24) $\quad \sup_{0\le t\le S} t|\nabla w(t)|^2 + \int_0^S t|g_2(t)|^2 dt \le C|\theta_o^n-\theta_o^m|^2,$

where $g_2=-w_t+B_2^t(u,w)\in A_2^t w$. Then it is easy to show that θ^n converges to the unique solution θ of $(3.12)_o$.

Proof of Fact III. Let $\theta_i\in K_{R,S}$ $(i=1,2)$, u_i be the solutions of (3.1) with $\hat\theta=\theta_i$ and let ψ_i be the solutions of (3.2) with $\hat u=u_i$. Then $\Theta=\theta_1-\theta_2$, $U=u_1-u_2$ and $\Psi=\psi_1-\psi_2$ satisfy

(3.25) $U_t + A_1^t U + P_B\{(u_1\cdot\nabla)U + (U\cdot\nabla)u_2 + (\bar u\cdot\nabla)U + (U\cdot\nabla)\bar u\} \ni P_B\eta g\Theta,\ U(0)=0,$

$$(3.26)\quad \Psi_t + A_2^t\Psi + (U\cdot\nabla)\psi_1 + (u_2\cdot\nabla)\Psi + (\bar{u}\cdot\nabla)\Psi + (U\cdot\nabla)\bar{\theta} = 0,\ \Psi(0) = 0.$$

In parallel with (3.5), we note

$$(3.27)\quad \left|\int_B (u\cdot\nabla)v\ w\ dx\right| \le \begin{cases} C\|u\|_4\|w\|_4\|\nabla v\| \le C\|u\|^{\frac12}\|w\|^{\frac12}\|u\|_{H^1}^{\frac12}\|w\|_{H^1}^{\frac12}\|\nabla v\| & N=2, \\ C\|u\|_6\|w\|_3\|\nabla v\| \le C\|u\|_{H^1}\|w\|_{H^1}^{\frac12}\|w\|^{\frac12}\|\nabla v\| & N=3. \end{cases}$$

Then multiplication of (3.25) by U and (3.27) give

$$(3.28)\quad \frac{d}{dt}\|U(t)\|^2 + \|\nabla U(t)\|^2 \le C\|U(t)\|^2 a(t) + |\eta|^2\|g\|_{\infty,T}^2|\theta(t)|^2,$$

where $a(t) = \begin{cases} 2(\|\nabla u_2(t)\| + \|\bar{u}(t)\|_{H^1})^2 + 1 & N=2, \\ 2(\|\nabla u_2(t)\| + \|\bar{u}(t)\|_{H^1})^4 + 1 & N=3. \end{cases}$

Hence, since $a\in L^1(0,S)$ by (3.7), we deduce

$$(3.29)\quad \|U(t)\|_{\infty,S}^2 + \int_0^S \|\nabla U(t)\|^2 dt \le C|\eta|^2\|g\|_{\infty,T}^2|\theta|_{\infty,S}\cdot S.$$

Moreover, multiplying (3.25) by $g_1 = -U_t - P_B\{(u_1\cdot\nabla)U + (U\cdot\nabla)u_2 + (\bar{u}\cdot\nabla)U + (U\cdot\nabla)\bar{u} - \eta g\theta\} \in A_1^tU$ and using (3.5), we obtain

$$\frac12\frac{d}{dt}\|\nabla U(t)\|^2 + \frac12\|g_1(t)\|^2 \le C(\|\nabla U\|^2 + |\eta|^2\|g\|_{\infty,T}^2|\theta|^2 + R(t)),$$

where $R(t) = R_1(u_1) + R_1(\bar{u}) + R_2(u_2) + R_2(\bar{u})$,

$R_1(v) = \|v\|^2\|v\|_{H^1}^2\|\nabla U\|^2,\ R_2(v) = \|v\|_{H^1}\|v\|_{H^2}\|U\|\|\nabla U\|$ if $N=2$,

$R_1(v) = \|v\|_{H^1}^4\|\nabla U\|^2,\ R_2(v) = \|v\|_{H^1}\|v\|_{H^2}\|\nabla U\|^2$ if $N=3$.

Hence, for the case N=3, by virtue of (A.α),(3.7) and Gronwall's inequality, we easily obtain

$$(3.30)\quad \|\nabla U\|_{\infty,S}^2 \le C\ |\theta|_{\infty,S}^2\cdot S\ .$$

As for the case N=2, since, by (3.29),

$$R_2(v(t)) \le \|v(t)\|_{H^1}^2\|\nabla U(t)\|^2 + C\|v(t)\|_{H^2}^2\cdot t\cdot|\theta|_{\infty,t}^2,$$

we deduce from (3.7) that

$$(3.31)\quad \|\nabla U\|_{\infty,S}^2 \le C|\theta|_{\infty,S}^2\cdot S^{2\mu},\ 0\le\mu\le \tfrac{1}{2}\ .$$

On the other hand, multiplication of (3.26) by Ψ yields

$$\tfrac{1}{2}\,\frac{d}{dt}|\Psi(t)|^2+|\nabla\Psi(t)|^2 \le \|U\|_3|\nabla\psi_1||\Psi|_6+\|U\|_3|\nabla\bar\theta||\Psi|_6,$$
$$\le \tfrac{1}{2}|\nabla\Psi|^2 + C\|U\|\|\nabla U\|(|\nabla\psi_1|^2+|\nabla\bar\theta|^2).$$

Since $|\nabla\psi_1|, |\nabla\bar\theta|\in L^2(0,S)$, it follows from (3.29),(3.30) and(3.31) that

$$|\Psi|_{\infty,S}^2 \le \begin{cases} C\ |\theta|_{\infty,S}^2\cdot S^{\frac{1}{2}+\mu} & N=2,\\ C\ |\theta|_{\infty,S}^2\cdot S & N=3.\end{cases}$$

Thus it is clear that $\mathcal{F}$ is a contraction for a sufficiently small S.

3.3. <u>Global Existence</u>.

3.3.1. <u>The case N=2</u>.

Multiplying (3.1) by u and (3.2) by θ and using (3.27), we have

$$(3.32)\ \tfrac{1}{2}\,\frac{d}{dt}\|u(t)\|^2+\|\nabla u(t)\|^2 \le C\|u\|\|\nabla u\|\|\nabla\bar u\| + \|u\|(\|F_1(t)\|+\|\eta g\theta\|),$$
$$\le \tfrac{1}{4}\|\nabla u\|^2+\|u\|^2(C\|\nabla\bar u\|^2+1)+\|F_1(t)\|^2+|\eta|^2\|g\|_{\infty,T}^2|\theta|^2,$$

$$(3.33)\ \tfrac{1}{2}\,\frac{d}{dt}|\theta(t)|^2+|\nabla\theta(t)|^2 \le C\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}|\nabla\bar\theta||\nabla\theta| + |F_2(t)||\theta|$$
$$\le \tfrac{1}{4}|\nabla\theta|^2+\tfrac{1}{4}\|\nabla u\|^2+C\|u\|^2|\nabla\bar\theta|^4+|\theta|^2+|F_2(t)|^2.$$

Adding together these inequalities, we easily deduce that there exists a number C_T depending only on $|\theta_o|, \|u_o\|, \bar{\theta}, \bar{u}$, and T but not on S such that

$$(3.34)\quad |\theta|_{\infty,S} + \|u\|_{\infty,S} + \int_0^S \|u(t)\|^2 dt \le C_T .$$

By mutiplication of (3.1) by $tg_1(t) = t(-u_t - B_1^t(u) + F_1(t) - P_B \eta\theta g) \in t\,A_1^t u(t)$ and estimate (3.6), we obtain

$$\tfrac{1}{2}\frac{d}{dt}(t\|\nabla u\|^2) + t\|g_1\|^2 \le \tfrac{t}{2}\|g_1\|^2 + Ct(\|\nabla u\|^2 + |\eta|^2\|g\|^2_{\infty,T}|\theta|^2 + \|F_1(t)\|^2)$$
$$+ C(\|u\|+1)t(\|\nabla u\|^4 + \|\bar{u}\|^4_{H^1} + \|\bar{u}\|^2_{H^2} + \|\bar{u}\|^4\|\bar{u}\|^4_{H^1}),$$

Hence, by (3.34) and Gronwall's inequality, we get

$$(3.35)\quad \sup_{0\le t\le S} t\|\nabla u(t)\|^2 \le C_T .$$

Thus these a priori bounds (3.34) and (3.35) together with the above local existence result assures that u, θ can be continued globally to [0,T] as solutions of (3.1) and (3.2).

3.3.2. <u>The case N=3</u>. Put $K_o = \sup(\|\nabla u\|/\|u\|, |\nabla\theta|/|\theta|)$, $K_1 = \sup(\|\nabla u\|/\|u\|_6, |\nabla\theta|/|\theta|_6)$, $\bar{K}_o = 1/\{1-\exp(-K_o^2/4)\}$ and take $\|\nabla\bar{u}\|$ and $\|g\|_{\infty,T}$ sufficiently small so that

$$(3.36)\quad |\nabla\bar{u}|^4 \le K_o^2 K_1^6/4,$$

$$(3.37)\quad 32\,\bar{K}_o^2 K_o^{-2} K_1^{-3}|\eta|^2 \sup_{0\le t\le T}\left(\int_{t-1}^t |\nabla\bar{\theta}(s)|^4 ds\right)^{\frac{1}{2}}\|g\|_{\infty,T} \le 1/2.$$

Then, by the same verification as for (3.32) and (3.33), we obtain

$$\tfrac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 \le K_1^{-\frac{3}{2}}\|\nabla u\|^{\frac{3}{2}}\|u\|^{\frac{1}{2}}\|\nabla\bar{u}\| + (\|F_1\| + |\eta|\,\|g\|_{\infty,T}|\theta|)\|u\|$$
$$\le \tfrac{3}{4}\|\nabla u\|^2 + \tfrac{1}{4}K_1^{-6}\|\nabla\bar{u}\|^4\|u\|^2 + K_o^2\|u\|^2/16 + 8K_o^{-2}(\|F_1\|^2 + |\eta|^2\|g\|^2_{\infty,T}|\theta|^2),$$

hence

$$(3.38)\quad \frac{d}{dt}\|u(t)\|^2 + K_o^2\|u(t)\|^2/4 \le 16K_o^{-2}(\|F_1\|^2 + |\eta|^2\|g\|_{\infty,T}^2|\theta|^2),$$

and similarly

$$(3.39)\quad \frac{d}{dt}|\theta(t)|^2 + K_o^2|\theta(t)|^2 \le K_1^{-3}\|u\|\|\nabla u\||\nabla\bar{\theta}|^2 + K_o^{-2}|F_2(t)|^2.$$

Then, from (3.38) and (3.39), we derive

$$(3.40)\quad \|u\|_{\infty,S}^2 + \frac{1}{8}\|\nabla u\|_{M,S}^2 \le \|u_o\|^2 + 16\,\bar{K}_o K_o^{-2}(\|F_1\|_{M,S}^2 + |\eta|^2\|g\|_{\infty,T}^2|\theta|_{\infty,S}^2),$$

$$(3.41)\quad |\theta|_{\infty,S}^2 \le |\theta_o|^2 + \bar{K}_o\{K_1^{-3}\|u\|_{\infty,S}\|\nabla u\|_{M,S}||\nabla\bar{\theta}|^2|_{M,S} + K_o^{-2}|F_2|_{M,T}^2\}.$$

Hence, by virtue of (3.37), we obtain an a priori upper bound for $|\theta|_{\infty,S}$ depending on $|\theta_o|, K_o, K_1, |F_2|_{M,T}$ but not on T. Furthermore, by (3.40), we see that $\|u\|_{\infty,S}$ and $\|\nabla u\|_{M,S}$ can be arbitrarily small if $\|u_o\|, \|F_1\|_{M,T}$ and $|\eta|\|g\|_{\infty,T}$ are taken small enough. Then the standard argument for Navier-Stokes equation can prove the statement of Theorem Ⅲ.

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