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Confluence and Completion of Membership Conditional TRS *

山田順之介 (NTT 基礎研究所)

Junnosuke Yamada
NTT Basic Research Laboratories
3-9-11, Midori-cho, Musashino, Tokyo 180 Japan

abstract

We propose a sufficient condition for the confluence of noetherian quasi-closed membership conditional term rewriting systems (MCTRS). The condition is the *critical pair lemma* for MCTRS. For that purpose, we introduce contextual rewriting which modifies contexts attached to terms, and we extend the notion of critical pair to the rules of such rewriting. By allowing modification of contexts, we can treat wider class of MCTRS than the previous work. As an application of the condition, we propose a completion algorithm for such systems. Additionally we use the completion algorithm for an inductionless induction like proof of a property of a recursively defined function.

1. Introduction

Most of our computation is based on equalities, and formalization and mechanization of such calculi have been investigated. In this direction, unconditional TRS appeared by regarding equalities as directed rewriting rules. Much research on TRS is stressed on its two principal characteristics, noetherian and confluent properties. When we try to apply the results to automated theorem proving, algebraic specification, program verification and transformation, we face to a difficulty. In real program, for example, the application of equalities is usually restricted by some conditions, then we come to a natural extension, conditional TRS in which rewriting rules have conditions for their usage. Such systems have already been investigated well, and we can find also results on the confluence of such systems.

But there is another approach for conditional TRS, membership conditional TRS whose rewriting rules are restricted by membership conditions on the variables in left hand sides of rules. Restrictions on types and values for variables in real programs can be expressed naturally using them. Moreover, such systems can describe a infinite number of rules in one. Thus membership conditional TRS will enable us to discuss automated theorem proving, specification, verification and transformation based on them.

Discussions on the confluence of unconditional TRS can be classified in two categories: one assumes left-linearity and non-overlappingness and another noetherian property. Research on conditional TRS has also similar two approaches and the reader can find more

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details in [4],[9],[10]. As for membership conditional TRS, the results in the former category are already known [13],[14] and we are going to show that in the latter. And we propose a completion algorithm for membership conditional TRS. This paper can treat more wide class of membership conditional TRS than the previous work [15] by introducing contextual rewriting which allows modification of context parts. Furthermore, we will apply our algorithm to show a property of famous 91-function, which is defined recursively and needs some inductive method to prove the property.

2. Term Rewriting Systems

In this section we briefly explain TRS and prepare necessary notions for the following sections. We assume that the reader is familiar with TRS and she or he can consult with the literatures (e.g., [2],[3],[5],[8]), if necessary.

A term set $T = T(F, V)$ is the set of first order terms composed of the elements in a set of function symbols F graded by arities and a denumerable set of variables V such that $F \cap V = \phi$. We use $Var(t)$ for the set of all the variables in a term t.

For any term t we can define its *occurrences* $\mathcal{O}(t)$, a subset of the set of sequences of positive integers $\mathbb{N}_+^*$, and *subterm* t/u of t at occurrence $u \in \mathcal{O}(t)$ as follows.

$$\begin{aligned} &\mathcal{O}(t) = \Lambda \text{ the empty sequence of } \mathbb{N}_+^* \text{ and } t/\Lambda \equiv t \text{ for } t \equiv x \in V, \\ &\mathcal{O}(t) = \{\Lambda\} \cup \{iu | i = 1, \cdots, n, u \in \mathcal{O}(t_i)\}, t/\Lambda \equiv t \text{ and } t/iu \equiv t_i/u \\ &\qquad \text{for } t = ft_1 \cdots t_n \quad \text{where } f \in F, t_i \in T. \end{aligned}$$

Next we define $t[u \leftarrow s]$ or simply $t[s]$ for $t, s \in T$ and $u \in \mathcal{O}(t)$ by

$$t[\Lambda \leftarrow s] \equiv s, \quad ft_1 \cdots t_n[iu \leftarrow s] \equiv ft_1 \cdots t_{i-1}(t_i[u \leftarrow s])t_{i+1} \cdots t_n.$$

A *substitution* θ is a map from V to $T(F, V)$ such that $\theta(x) \equiv x$ almost everywhere.

A *rewriting rule* on T is a pair of two terms (l, r) with $Var(l) \supset Var(r)$ and $l \notin V$. We denote a set of rewriting rules by $\triangleright$, and write $l \triangleright r$ iff $(l, r) \in \triangleright$. A term t *reduces to* a term t' at occurrence $u \in \mathcal{O}(t)$ by a rewriting rule $l \triangleright r$ iff $t \equiv s[u \leftarrow l\theta]$, $t' \equiv s[u \leftarrow r\theta]$ for some $s \in T$, substitution θ, and occurrence $u \in \mathcal{O}(t)$ such that $t/u \notin Var(t)$. The relation of the two terms is indicated by $t \to t'$ and the subterm t/u is called a *redex* of the rule in t.

We define term rewriting system.

Definition 2.1. (Term Rewriting System)

A TRS is a structure $(T, \to)$ with an object set T and a binary relation $\to$ defined by a set of rewriting rules $\triangleright$ on T.

We express by $\to^*$ the transitive reflexive closure of $\to$. A term t is said to be a *normal form* iff there is no t' such that $t \to t'$. A term t' is called a *normal form* of t iff $t \to^* t'$ and t' is a normal form and denoted by $t\downarrow$. Two terms t_1 and t_2 *converge* or are *convergent* iff there is a term s such that $t_1 \to^* s$ and $t_2 \to^* s$.

Two rules $l_i \triangleright r_i$ for $i = 1,2$ with no common variables in a TRS are *overlapping* iff $l_i\theta_i/u \equiv l_j\theta_j$ for some $\theta_i, \theta_j, u \in \mathcal{O}(t)$ such that $l_i/u \notin V$. Hereafter any two rules are assumed to have no common variables, if not stated.

We can define a critical pair of two overlapping rules.

Definition 2.2. (Critical Pair)
A pair of terms $\langle P, Q\rangle$ is a *critical pair* of two rules $l_i \triangleright r_i$ for $i = 1,2$ overlapping in $u \in \mathcal{O}(l_1)$ is:

$$P \equiv l_1\theta[u \leftarrow r_2\theta], \quad Q \equiv r_1\theta$$

where θ is the most general unifier of $l_1/u \notin V$ and l_2.

The following two notions characterize TRS.

Definition 2.3. (Noetherian)
A TRS $R = (T, \rightarrow)$ is *noetherian* iff every reduction in R terminates, i.e., there is no infinite reduction sequence as $t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow \cdots$ where $t_i \in T$.

Definition 2.4. (Confluence and Local Confluence)
A TRS $R = (T, \rightarrow)$ is *confluent* iff

$$\forall u, v, w \in T[u \rightarrow^* v, u \rightarrow^* w \Rightarrow \exists u' \text{ such that } v \rightarrow^* u', w \rightarrow^* u']$$

and *locally confluent* iff

$$\forall u, v, w \in T[u \rightarrow v, u \rightarrow w \Rightarrow \exists u' \text{ such that } v \rightarrow^* u', w \rightarrow^* u'].$$

These two properties have been of our chief concern, because noetherian property guarantees the existence of normal forms, and confluence does uniqueness of normal forms provided existence of them. A TRS equipped with both properties is said *complete*. In such systems, every term has necessarily an unique normal form.

For noetherian unconditional TRS the following results on the confluence are well-known.

Lemma 2.5. (Critical Pair Lemma)
A noetherian TRS R is locally confluent if and only if every critical pair of R converges.

We note the next lemma for general noetherian relations.

Lemma 2.6.
A noetherian relation is confluent if and only if it is locally confluent.

Combining these two lemmas, the next theorem on the confluence of unconditional noetherian TRS holds.

Theorem 2.7.
A noetherian TRS R is confluent if and only if every critical pair of R converges.

3. Membership Conditional Term Rewriting Systems

We introduce a kind of conditional TRS, membership conditional TRS.

Definition 3.1. (c-Term, MC-Rule)

A *c-term* is a term with membership conditions on the variables in the term:

$$t : (x_1, \cdots, x_n) \in S_1 \times \cdots \times S_n$$

where $\{x_1, \cdots, x_n\} = Var(t)$ and $S_i \subset T$ for all i, and written simply as $t : c$. We call $c = (x_1, \cdots, x_n) \in S_1 \times \cdots \times S_n$ the *context* of the c-term. A *MC-rule* $l \triangleright r : c$ is a rewriting rule $l \triangleright r$ with membership conditions c on the variables in l.

We say that a term t reduces to a term t' by a MC-rule $l \triangleright r : (x_1, \cdots, x_n) \in S_1 \times \cdots \times S_n$ in a membership conditional TRS, when

$$t \equiv s[l\theta], \quad t' \equiv s[r\theta] \text{ for some } s \in T, \text{ substitution } \theta \text{ and } x_1\theta \in S_1, \cdots, x_n\theta \in S_n.$$

Definition 3.2. (Membership Conditional TRS)

A *membership conditional TRS* is a term rewriting system defined by a set of MC-rules.

An example of membership conditional TRS and its reductions are shown below.

Example 3.3.

Let $F = \{eq, d, +, s, 0\}$ and $F' = \{+, s, 0\}$. Next membership conditional TRS R defines the addition $+$, the double d, and eq on the set of natural numbers $\mathbb{N} = T(\{s, 0\})$.

$$R : \begin{cases} x + 0 \triangleright x : x \in T \\ x + s(y) \triangleright s(x + y) : (x, y) \in T^2 \\ d(x) \triangleright x + x : x \in T(F') \\ eq(x, x) \triangleright x : x \in T(F') \end{cases}$$

In this system, we have the following reduction sequence:

$$eq(d(0), d(0)) \to eq(0 + 0, d(0)) \to eq(0 + 0, 0 + 0) \to 0 + 0 \to 0.$$

Note that a direct reduction $eq(d(0), d(0)) \to d(0)$ is impossible by the third rule in R since $d(0) \notin T(F')$.

Definition 3.4. (Closed, Quasi-Closed, Terminating and Normal)

A set of terms $S \subset T$ is *closed* iff $\forall s \in S \forall t \in T[s \to t \Rightarrow t \in S]$, *quasi-closed* iff $s \downarrow \in S$ for all $s \in S$, *terminating* iff there is no infinite reduction sequence $s \to s' \to s'' \to \cdots$ for $\forall s \in S$, and *normal* iff S consists of normal forms. A membership conditional TRS R is *closed*, *quasi-closed*, *terminating*, and *normal* iff every set not equal to term set T which appears in membership conditions of its rules is closed, terminating, normal and quasi-closed respectively. We remark that a normal membership conditional TRS is closed and terminating.

Membership conditional TRS might be paradoxical and NOT well-defined as other conditional TRS. The reader can refer to [14] about the point. We also assume that membership conditional TRS treated are all well-defined.

On the confluence of NOT-noetherian membership conditional TRS, there are some results in [13],[14]. Now we seek for some criterion on the confluence providing noetherian property as in the case of the other TRS (cf. [1], [4], [8]).

4. Contextual Rewriting

First we introduce contextual rewriting which differs from the one in [16], and we suite it to membership conditional TRS. The notion is prerequisite to discuss critical pairs of MC-rules. Furthermore we extend the previous work [15] by allowing modification of context parts of c-terms even in a restricted manner.

Definition 4.1. (c-Reduction)
A c-term $t : c$ is *c-reducible* by a MC-rule $l \triangleright r : (x_1, \cdots, x_n) \in S_1 \times \cdots \times S_n$ iff some subterm $t' \equiv t/u$ at occurrence u of t is $l\theta'$ for some substitution θ' and $x_i\theta' \in S_i$ for $i = 1, \cdots, n$ hold under c. Then $t : c$ *c-reduces to* $s : c \equiv t[u \leftarrow r\theta'] : c$ and we denote $t : c \rightarrow_c s : c$. In the above definition, as each $x_i\theta'$ includes variables restricted by context c, we have to verify that $x_i\theta' \in S_i$. The transitive reflexive closure of $\rightarrow_c$ is denoted by $\rightarrow_c^*$. A c-term $t : c$ is a *c-normal form* iff there is no $t' : c$ such that $t : c \rightarrow_c t' : c$, and $t' : c$ is a *c-normal form* of $t : c$ iff $t : c \rightarrow_c^* t' : c$, and $t' : c$ is a c-normal form. Two c-terms $t_1 : c$ and $t_2 : c$ with a common context *c-converge* iff there is a c-term $s : c$ such that $t_1 : c \rightarrow_c^* s : c$ and $t_2 : c \rightarrow_c^* s : c$.

This is a kind of contextual rewriting that preserves contexts written by membership conditions and we show an example.

Example 4.2.
By a MC-rule $f(x) \triangleright g(x) : x \in \mathbb{N}$, we have a c-reduction:

$$h(f(s^2(y))) : y \in \mathbb{N} \rightarrow_c h(g(s^2(y))) : y \in \mathbb{N}.$$

In this case, we have to check $s^2(y) \in \mathbb{N}$ under $y \in \mathbb{N}$ and succeed.

We formalize the relation between terms and c-terms.

Definition 4.3. (Instance of c-Term, Associated c-Term)
A term $t\theta$ is called an *instance* of a c-term $t : c$, iff $x\theta$ satisfies the condition c for any variable $x \in Var(t)$. Conversely, we call $t : c$ an *associated c-term* of $t\theta$. We call a set of c-terms $T_c = \{t : c \mid t \in T \text{ with membership conditions } c\}$ as an *associated c-term set* of T.

We have to establish a correspondence between terms and c-terms and one between TRS and c-TRS.

Lemma 4.4. (Existence of Associated c-Term)
For any term t there is some associated c-term $t' : c'$ and $t \equiv t'\theta$.

Proof.
Clear by the inclusion map $t \longmapsto t : x_1 \in \{x_1\}, \cdots, x_n \in \{x_n\}$. □

Lemma 4.5.
If $t : c$ c-reduces to $s : c$ and $t\theta$ is an instance of $t : c$, then there is an instance $s\theta$ of $s : c$ such that $t\theta \to s\theta$. That is, the diagram below is commutative:

$$\begin{array}{ccc} t : c & \longrightarrow_c & s : c \\ \downarrow & & \downarrow \\ t\theta & \longrightarrow & s\theta \end{array}$$

Proof.
Let $t : c \to_c s : c$ by applying a c-rule $l \triangleright r : \tilde{c}$ to a redex $t/u : c$ of $t : c$. If the rule is applicable also to $t\theta/u$, then we have the below commutative diagram:

$$\begin{array}{ccc} t : c \equiv t[l\theta'] : c & \longrightarrow_c & s[r\theta'] : c \equiv t : c \\ \downarrow & & \downarrow \\ t\theta \equiv t\theta[l\theta\theta'] & \longrightarrow & s\theta[r\theta\theta'] \equiv t\theta \end{array}$$

Then it remains only to show that the rule is applicable to $t\theta/u$. Let $y \in S$ be a condition in $\tilde{c}$, then $y\theta' \in S$ under $c = (x_1, \cdots, x_n) \in S_1 \times \cdots \times S_n$. From the definition of instance $(x_1\theta, \cdots, x_n\theta) \in S_1 \times \cdots \times S_n$, we have $y\theta'\theta \in S$, i.e., $t\theta/u$ is also a redex of the rule. $\square$

By lemmas 4.4 and 4.5, we can define the ***associated cTRS*** $(T_c, \to_c)$ of TRS $(T, \to)$.
Definition 4.6.
For a TRS $R = (T, \to)$ we have a set of associated c-terms T_c and a c-reduction relation $\to_c$, and a TRS $(T_c, \to_c)$ called the ***associated cTRS*** of R. Moreover we can necessarily define the associated cTRS for any TRS.

Based on the notion of c-reduction, we can define a critical pair of two MC-rules. Before that, we have to clarify the notion of overlapping in the case of cTRS.

Definition 4.7. (c-Overlapping)
Two MC-rules with no common variables

$$l_1 \triangleright r_1 : (x_1, \cdots, x_m) \in S_1 \times \cdots \times S_m \quad \text{and}$$
$$l_2 \triangleright r_2 : (x_{m+1}, \cdots, x_{m+n}) \in S_{m+1} \times \cdots \times S_{m+n}$$

are *c-overlapping in root* iff
(1) there is the most general unifier $\theta = mgu(l_1, l_2) \neq \phi$ of l_1 and l_2, and
(2) for every substitution $x_i/\theta(x_i) \in \theta$, $\theta(x_i) \in S_i$ under $(x_{i_1}, \cdots, x_{i_k}) \in S'_{i_1} \times \cdots \times S'_{i_k} \subset S_{i_1} \times \cdots \times S_{i_k}$ where $\{x_{i_1}, \cdots, x_{i_k}\} = Var(\theta(x_i))$.
They are *c-overlapping in occurrence* $u \in \mathcal{O}(l_1)$ iff $l_1/u \notin V$ and l_2 are c-overlapping in root.

Now we can introduce c-critical pair of two c-overlapping rules.

Definition 4.8. (c-Critical Pair)
Let

$$l_1 \triangleright r_1 : (x_1, \cdots, x_m) \in S_1 \times \cdots \times S_m \quad \text{and}$$
$$l_2 \triangleright r_2 : (x_{m+1}, \cdots, x_{m+n}) \in S_{m+1} \times \cdots \times S_{m+n}$$

be two MC-rules c-overlapping in occurrence $u \in \mathcal{O}(l_1)$. The *MC-critical pair* $\langle P, Q\rangle : c$ of the two MC-rules in $u \in \mathcal{O}(l_1)$ is

$$P \equiv l_1[u \leftarrow r_2\theta], \quad Q \equiv r_1\theta, \quad \text{and} \quad c \equiv (x_{j_1}, \cdots, x_{j_k}) \in S'_{j_1} \times \cdots \times S'_{j_k}$$

where $\{x_{j_1}, \cdots, x_{j_k}\} = Var(P) \cup Var(Q)$ and $S'_{j_\mu} \subset S_{j_\mu}$ which makes two rules c-overlapping for $\mu = 1, \cdots, k$.

Remark 4.9.
The notions of c-overlapping and c-critical pair are also effective in contextual membership conditional TRS in which modification of conditions is allowed.

We will try to find a c-critical pair of the following two rules:

$$f(f(y)) \triangleright h(y) : y \in g(T) \cup h(T) \quad \text{and}$$
$$f(g(z)) \triangleright g(z) : z \in T.$$

We have a substitution $y/g(z)$ to made the two rules c-overlapping. Then we have to find some subcondition of $z \in T$ from $g(z) \in g(T) \cup h(T)$. In this example, we can immediately find such a condition $z \in T$ and a c-critical pair $\langle f(h(z)), h(g(z))\rangle : z \in T$.

Contextual rewriting as c-reduction and related notions are too restricted. For example, we try to c-reduce a single c-term $f(x) : x \in \mathbb{N}$ by two rules:

$$f(x) \triangleright 0 : x \in \{\text{even}\} \quad \text{and} \quad f(x) \triangleright 1 : x \in \{\text{odd}\}$$

But we find that $f(x) : x \in \mathbb{N}$ is c-irreducible. Then we introduce extended reduction which can handle above example.

Definition 4.10. (Splitting)
A context $c = (x_1, \cdots, x_k) \in S_1 \times \cdots \times S_k$ *splits* into $c_1, \cdots, c_n$ iff some of sets in it, for example, $S_1, \cdots, S_{k_0}$ with $k_0 \leq k$ are disjoint unions of several sets, i.e., $S_i = \sqcup S_i^{(j_i)}$. We denote this by $c = c_1 \sqcup \cdots \sqcup c_n$ and $c_i = S_1^{(j_1)} \sqcup \cdots \sqcup S_{k_0}^{(j_{k_0})} \sqcup S_{k_0+1} \sqcup \cdots \sqcup S_k$.

A c-term $t : c$ *sc-reducible* iff there is a splitting $c = c_1 \sqcup \cdots \sqcup c_n$ and $m \leq n$ such that $t : c_j \rightarrow_c t_j : c_j$ for all $j \leq m$ and $t : c_j$ is c-irreducible for all $j > m$. Regarding a single c-term $t : c$ with splitting $c = \sqcup_{j \leq n} c_j$ as a set of c-terms, we have a *sc-reduction* $\{t : c\} \rightarrow_{sc} \{t'_j : c_j\}$ when either $t : c_j \rightarrow_c t'_j : c_j$ for $j = 1, \cdots, n-1$ and $t : c_n \equiv t'_n : c_n$ is c-irreducible or $t : c_j \rightarrow_c t'_j : c_j$ for $j = 1, \cdots, n$. Two c-terms $t' : c$ and $t'' : c$ with a common context c *sc-converge* iff c splits into $c_1 \sqcup \cdots \sqcup c_n$ such that every pair $t' : c_i$ and $t'' : c_i$ c-converge.

Definition 4.11.
The *scTRS* of a membership conditional TRS $R = (T, \rightarrow)$ is $R_{sc} = (2^{T_c}, \rightarrow_{sc})$ constructed from its associated cTRS $R_c = (T_c, \rightarrow_c)$ by allowing splitting.

As is easily seen, every scTRS can be regarded as a cTRS, because every term and reduction in scTRS correspond to several terms and reduction sequences in cTRS.

Using these notions we can execute a kind of reduction in the last example:

$$\{f(x) : x \in \mathbb{N}\} \rightarrow_{sc} \{0 : x \in \{\text{even}\}, 1 : x \in \{\text{odd}\}\}.$$

Unfortunately this extension draws another difficulty, that is, even if a membership conditional TRS R is noetherian, its scTRS R_{sc} might be no longer noetherian. Then it is impossible to determine whether or not critical pairs are convergent. We consider the following noetherian membership conditional TRS:

$$R : \{\, f(s(x)) \triangleright f(x) : x \in \mathbb{N}.$$

We have the next infinite sequence:

$$\begin{array}{ccccccc} f(x) : x \in \mathbb{N} & \rightarrow & f(x') : x' \in \mathbb{N} & \rightarrow & f(x'') : x'' \in \mathbb{N} & \rightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ f(0) : x = 0 & & f(0) : x = 1 & & f(0) : x = 2 & & \end{array}$$

where $x = s(x') = s(s(x'')) = \cdots$. We restrict ourselves to the cases in which infinite sequences as above do not occur. In fact, when sets in membership condition not equal to T are finite, such pathological reductions do not occur.

5. Confluence of Membership Conditional Term Rewriting Systems

Next lemma is the critical pair lemma for cTRS.

Lemma 5.1.
Let $R_c = (T_c, \rightarrow_c)$ be a noetherian quasi-closed cTRS. If every c-critical pair of R_c converges, then R_c is locally confluent.

Proof.
Similar to the unconditional case and the other conditional cases. □

The following is our key lemma which allows us to interpret the locally confluence of cTRS into that of TRS.

Lemma 5.2.
A TRS $R = (T, \rightarrow)$ is locally confluent, if its associated cTRS $R_c = (T_c, \rightarrow_c)$ is locally confluent.

Proof.
Let t_1, t_2 be two terms reduced from a single term t in R, then there are three associated c-terms $s : c$, $s_1 : c$ and $s_2 : c$ of t, t_1 and t_2 respectively such that $s_1 : c$, $s_2 : c$ are

c-reduced from $s : c$ in R_c.

$$
\begin{array}{ccccccc}
 & & & s : c & & & \\
 & & & \downarrow & & & \\
 & \swarrow & & t & & \searrow & \\
 & & \swarrow & & \searrow & & \\
s_1 : c & \longrightarrow & t_1 & & t_2 & \longleftarrow & s_2 : c \\
 & & \searrow * & & * \swarrow & & \\
 & \searrow * & & t' & & * \swarrow & \\
 & & & \uparrow & & & \\
 & & & s' : c & & &
\end{array}
$$

By the hypothesis there is a c-term $s' : c$ such that $s_1 : c \to_c^* s' : c$ and $s_2 : c \to_c^* s' : c$. Thus we have an instance t' of $s' : c$ such that $t_1 \to^* t'$ and $t_2 \to^* t'$ by lemma 4.5. $\square$

Now we can give a criterion of the confluence assuming noetherian property for membership conditional TRS as in the unconditional and other conditional cases.

Theorem 5.3.
Let R be a noetherian quasi-closed membership conditional TRS. If every c-critical pair of R c-converges, then R is confluent.

Proof.
We have the associated cTRS R_c of R. As its every c-critical pair converges, R_c is locally confluent by lemma 5.1. Then R is also locally confluent using lemma 5.2, moreover confluent for its noetherian property and lemma 2.6. $\square$

Note that a c-reduction can be regarded as sc-reductions and a c-critical pair is also a critical pair of contextual rewriting rules allowing splitting. Then the following lemma is clear.

Lemma 5.4.
If every c-critical pair in R_{sc} sc-converges, every c-critical pair c-converges in R_c.

It is clear that every c-critical pair in R_c can be regarded also as the one in R_{sc}. Then using this lemma and theorem 5.3, we can easily have the following theorem.

Theorem 5.5. (Main Theorem)
Let R be a noetherian quasi-closed membership conditional TRS. If every c-critical pair in R sc-converges, then R is confluent.

Based on the above theorem we can design a completion algorithm as in the unconditional case ([6], [11]) and the other conditional cases (e.g., [10]).

Let a set of quasi-closed membership conditional equalities E and some reduction ordering $\gg$ be given. We assume that selection of equality $m = n : c$ from E satisfies the *fairness hypothesis* in [6]. The hypothesis ensures every equality E will be selected within a finite number of steps in the completion algorithm below.

Completion Algorithm

E : a set of quasi-closed MC-equalities (given)
R : a set of MC-rules (initially $= \phi$)

Loop while $E \neq \phi$ do
if $E = \phi$ then $return(R)$;;; Stops with success, R is complete.
$f := m = n : c$;;; A candidate of a new rule, chosen from E.
$r := \{l_i \triangleright r_i : c_i\}$
;;; New rules, l_i, r_i are c-normal forms of $m : c_i$, $n : c_i$
;;; by the current rule set R and $l_i \gg r_i$. If c-normal forms
;;; of $m : c_i$ and $n : c_i$ are IN-comparable by $\gg$, then stops with failure.
$R' := \{l' \triangleright r' : c' \in R \mid l' : c'$ or $r' : c'$ sc-reducible by some $l_i \triangleright r_i : c_i \in r\}$
$R'_{eq} := \{l' = r' : c' \mid l' \triangleright r' : c' \in R'\}$
$R := R + r - R'$
$E := E - \{f\} + R'_{eq} + CP\langle R, r\rangle$
;;; $CP\langle R, r\rangle$ is all the c-critical pairs between the rules in new R and r.

Now we show a theorem on the completeness of above algorithm and its proof.

Theorem 5.6.
For a given set of quasi-closed MC-equalities E and a reduction ordering $\gg$, when above algorithm stops and we have a membership conditional TRS R, then R is complete, that is, noetherian and confluent and moreover $=_E \equiv \sim_R$. Here $=_E$ and $\sim_R$ denote equivalence relations generated by $=$ of E and $\rightarrow$ of R respectively.

Proof.
We indicate $E, R, \cdots$ in i-th loop with suffix i as $E_i, R_i, \cdots$. As $=_{E_{i+1}} \cup \sim_{R_{i+1}} \supset =_{E_i} \cup \sim_{R_i}$ clearly and its converse is also true by $=_{CP\langle R_i, r_i\rangle} \subset =_{R_i} \cup =_{f_i}$, we have $=_E \equiv \sim_R$. When $E = \phi$, R is locally confluent because there is no critical pair, and noetherian by the ordering used in the algorithm. □

This completion algorithm can be used as a proof method similar to inductionless induction in the unconditional case in [7]. Under axioms described by a complete membership conditional TRS R satisfying some property alike definition principle, we can prove whether or not $m = n : c$ is a theorem of R by trying to complete $R \cup \{m = n : c\}$. If the completion stops with success the equality is a theorem and if stops with failure not a theorem. Proof of this method in our case is similar to the unconditional case. Now the next example is examined.

Example 5.7. (91-function)
The famous 91-function f in [12] is recursively defined by

$$f(x) = \text{ if } x \leq 100 \text{ then } f(f(x + 11)) \text{ else } x - 10$$

and has a property

$$f(x) = 91 \text{ for all } x \leq 100.$$

We can reformulate the definition of f by the following membership conditional equalities (1) and (2), the property by (3). We indicate membership conditions by equalities or inequalities which define the sets in them, e.g., $x \leq 100$ means $x \in \{n \in \mathbb{N} \mid n \leq 100\}$.

$$E\colon \begin{cases} f(x) = x - 10 \quad : 101 \leq x & (1) \\ f(x) = f(f(x+11)) \quad : x \leq 100 & (2) \\ f(x) = 91 \quad : x \leq 100 & (3) \end{cases}$$

Now we prove the property (3) using our completion algorithm for membership conditional TRS. That is, we show that our completion generates a complete set of MC-rules from the MC-equality system E. Before completing above E, we notice that our algorithm alos succeeds in completing $\{(1),(2)\}$. From now on we denote current MC-rule set by R and assume the order to be $f \gg s \gg 0$ where s is the successor function and $\mathbb{N} = T(\{s,0\})$.

We have the following membership conditional TRS R choosing (1) and (3) from E or reverse order.

$$R\colon \begin{cases} f(x) \triangleright x - 10 \quad : 101 \leq x & (1') \\ f(x) \triangleright 91 \quad : x \leq 100 & (3') \end{cases}$$

Then we try to add the last equality (2) in E as a news rule, then the equality is reduced as below by $(1')$ and $(3')$.

$$\begin{array}{ccccccc} f(x) : x \leq 100 & = & f(f(x+11)) : x \leq 100 & \rightarrow & f(x+1) : 90 \leq x \leq 100 & \rightarrow & x - 9 : x = 10 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 91 : x \leq 100 & & f(91) : x \leq 89 & & 91 : 90 \leq x \leq 99 & & 91 : x = 100 \\ & & \downarrow & & & & \\ & & 91 : x \leq 89 & & & & \end{array}$$

This shows that the both sides of equality (2) reduce to the same, there remains no equality in E and our completion succeeds.

Thus we have proved the intended property of 91-function, by describing it by a membership conditional TRS and applying our completion algorithm to the system. If we do not utilize membership conditional TRS, then we have to define f by the following unconditional TRS which includes many rules and takes much effort for completion.

$$\begin{cases} f(s^{11}(0)) \triangleright f(0) \\ f(s^{11}(s(0))) \triangleright f(s(0)) \\ f(s^{11}(s(s(0)))) \triangleright f(s(s(0))) \\ \quad\vdots \\ f(s^{11}(s^{100}(0))) \triangleright f(s^{91}(0)) \\ f(s^{101}(x)) \triangleright s^{91}(x) \end{cases}$$

Comparing with this method, our method is simpler and more efficient as it has much less number of equalities and rules.

7. Conclusion

We investigated the confluence of noetherian quasi-closed membership conditional TRS. For that purpose we introduced contextual terms, contextual rewriting which modifies contexts of them, and contextual critical pairs. Using such notions, it was shown that a noetherian quasi-closed membership conditional TRS is confluent if its every contextual critical pair sc-converges. Based on this criterion, we proposed a completion procedure for membership conditional TRS. Moreover the completion was used in our inductionless induction like proof of a property of famous 91-function.

As membership conditional TRS is a natural method to treat equality systems and programs expressed by equalities, so our algorithm is applicable to automated theorem proving, verification and transformation of programs.

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