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Closure properties of ω-languages under morphism and inverse morphism

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0. Introduction

Since the study of ω-regualar languages was started by Büchi[1], many classes of ω-regular languages have been investigated [6, 7]. These classes are mainly defined by automata with various types of accepting conditions. Among others, the following four types (E), (E'), (I), and (I') of accepting conditions are extensively studied. Let $\mathcal{A}$ be a finite automaton, w be an ω-word, and r be the run of $\mathcal{A}$ on w.

(E) An accepting state of $\mathcal{A}$ appears in r.

(E') All states appearing in r are accepting states.

(I) Accepting state appears infinitely many times in r.

(I') All states appearing infinitely many times in r are accepting states.

Recently the following two types of accepting conditions and the classes defined thereby are studied [5, 10]:

(L) All accepting states appear infinitely many times in the run of $\mathcal{A}$ on w.

(L') An accepting state appears at most finitely many times in the run of $\mathcal{A}$ on w.

The class of ω-languages defined by deterministic finite automata with the acceptance of type E (E', I, I', L or L') is denoted by $\mathbb{E}$ ($\mathbb{E}'$, $\mathbb{I}$, $\mathbb{I}'$, $\mathbb{L}$, $\mathbb{L}'$, $\mathbb{L}$ or $\mathbb{L}'$ resp.).

As is well known in formal language theory, many important familes are trios. Trios and full trios are closed under many other operations. (See, for example, [2].) Especially, morphism and inverse morphism are important, since there are some characterizations for families of languages without intersection with regular sets [9, 11].

In this paper we deal with ε-free morphisms and inverse morphisms. Especially we pay our attention to the classes close under ε-free morphisms and ω-free inverse morphisms. In section 2 we provide the closure property of these classes under ω-free morphisms, and in section 3 under inverse morphisms. In section 4 we consider a "duo", a class of ω-languages closed under ε-free morphisms and inverse ω-free morphisms. We first investigate the principality for the three duos $\mathbb{R}_\omega$, $\mathbb{I}'$ and $\mathbb{E}'$ In [9] it has alredy been proved that $\mathbb{R}_\omega$ and $\mathbb{E}'$ are principal, i.e., each language in $\mathbb{E}'$ ($\mathbb{R}_\omega$) is obtained from a particular ω-language in $\mathbb{E}'$ ($\mathbb{R}_\omega$, resp.) by finitely many applications of these two kinds of operations. We show that $\mathbb{I}'$ is also principal. Furthermore we present representation theorems for the three duos. Finally we consider for each of $\mathbb{E}$, $\mathbb{E}'$, $\mathbb{I}$, $\mathbb{I}'$, $\mathbb{L}$ and $\mathbb{L}'$, the smallest duo that contains it.

1. Preliminaries

Let Σ be an alphabet. Σ^* denotes the set of all finite words over Σ, and Σ^ω denote the set of all ω-words over Σ, i.e., the set of all mappings $\alpha : \{0, 1, 2, \ldots\} \to \Sigma$. Let $\Sigma^\infty = \Sigma \cup \Sigma^\omega$. An ω-word is written by $\alpha = a_0a_1\ldots$ where $a_n = \alpha(n)$ $(n = 0, 1, 2, \ldots)$. We call a subset of Σ^* (Σ^ω, resp.) a <u>language</u> (<u>ω-language</u>) over Σ. We define the ω-power of the languages L as $L^\omega = \{w_0w_1w_2\ldots \mid w_0, w_1, \ldots \in L - \{\varepsilon\}\}$. Here ε stands the empty word.

A <u>deterministic automaton</u> (DA, for short) $\mathcal{A}$ over Σ is a 5-tuple $\mathcal{A} = \langle S, \Sigma, \delta, s_0, F\rangle$, where S is a finite set of states, Σ is an alphabet, $\delta : S \times \Sigma \to S$ is a next state function, $s_0 \in S$ is an initial state, and $F \subseteq S$ is a set of accepting states. Then the <u>run</u> Run($\mathcal{A}$, α) of automaton $\mathcal{A}$ on an ω-word α is an ω-word $q_0q_1\ldots \in Q^\omega$ such that

$$q_0 = s_0 \text{ and } q_{n+1} = \delta(q_n, \alpha(n)) \quad (n=0, 1, 2, \ldots).$$

For a run r, let

$$Ex(r) = \{ q \in Q \mid q = q_n \text{ for some } n \}, \text{ and}$$

$$Inf(r) = \{ q \in Q \mid q = q_n \text{ for infinitely many } n \}.$$

Now we define the following six types of acceptances of the automaton $\mathcal{A}$ for ω-languages:

$$E'(\mathcal{A}) = \{ \alpha \in \Sigma^\omega \mid Ex(Run(\mathcal{A}, \alpha)) \subseteq F \}$$

$$I(\mathcal{A}) = \{ \alpha \in \Sigma^\omega \mid Inf(Run(\mathcal{A}, \alpha)) \cap F \neq \phi \}$$

$$I'(\mathcal{A}) = \{ \alpha \in \Sigma^\omega \mid Inf(Run(\mathcal{A}, \alpha)) \subseteq F \}$$

$$L(\mathcal{A}) = \{ \alpha \in \Sigma^\omega \mid F \subseteq Inf(Run(\mathcal{A}, \alpha)) \}$$

$$L'(\mathcal{A}) = \{ \alpha \in \Sigma^\omega \mid F \not\subseteq Inf(Run(\mathcal{A}, \alpha)) \}$$

These acceptances are called E-, E'-, I-, I-, and L'-acceptance resp., and the class of ω-languages (over Σ) of the form $E(\mathcal{A})$, $E'(\mathcal{A})$, $L(\mathcal{A})$, $L'(\mathcal{A})$, $I(\mathcal{A})$, $I'(\mathcal{A})$, respectively) are denoted by $\mathbb{E}$, ($\mathbb{E}'$, $\mathbb{L}$, $\mathbb{L}'$, $\mathbb{I}$, $\mathbb{I}'$).

A <u>nondeterministic automaton</u> (NDA, for short) is a 5-tuple $<S, \Sigma, \delta, s_0, F>$, where S, Σ, s_0, and F have the same meaning as for a deterministic automaton, but δ is a mapping $S \times \Sigma \to \mathbb{P}(S)-\{\phi\}$. For an NDA $\mathcal{A}$ and an ω-word w, let Run($\mathcal{A}$, w) be the set of runs of $\mathcal{A}$ on w. Simularly as for the deterministic case, we define the six ω-languages $NE(\mathcal{A})$, $NE'(\mathcal{A})$, $NI(\mathcal{A})$, $NI'(\mathcal{A})$, $NL(\mathcal{A})$, and $NL'(\mathcal{A})$ accepted by $\mathcal{A}$.

We denote by $\mathcal{N}\mathbb{E}$ ($\mathcal{N}\mathbb{E}'$, $\mathcal{N}\mathbb{I}$, $\mathcal{N}\mathbb{I}'$, $\mathcal{N}\mathbb{L}$, $\mathcal{N}\mathbb{L}'$) the class of ω-languages of the form $NE(\mathcal{A})$($NE'(\mathcal{A})$, $NI(\mathcal{A})$, $NI'(\mathcal{A})$, $NL(\mathcal{A})$, $NL'(\mathcal{A})$ respectively). For inclusinon relations among these classes, see [5].

Let Σ and Δ be two alphabets. A morphism $h: \Sigma^* \to \Delta^*$ is said to be ε-free if $h(\Sigma) \subseteq \Delta^+$, where $\Delta^+ = \Delta^*-\{\varepsilon\}$.

Let $\mathcal{L}$ be a class of ω-languages. $\mathcal{L}$ is called a <u>duo</u> if it is closed under ε-free morphism and inverse ε-free morphism. For a class $\mathcal{L}$ of ω-languages, the smallest duo that contains $\mathcal{L}$ is denoted by $\mathcal{D}(\mathcal{L})$. $\mathcal{L}$ is called a <u>principal</u> duo generated by an ω-language X, denote by $\mathcal{D}(X)$, if it is the smallest duo that contains X. The ω-language X is called a <u>generator</u> of $\mathcal{L}$.

2. Closure under morphisms

Morphisms given in this section are ε-free.

<u>Theorem 2.1 [3, 8].</u> The classes $\mathbb{R}_\omega$ and $\mathbb{E}'$ are closed under

morphism.

<u>Theorem 2.2</u>. The class $\mathbb{I}'$ is closed under morphism.

(Proof) Let $L = I'(\mathcal{A})$ for a DA $\mathcal{A} = <Q, \Sigma, \delta, q_0, F>$, and $h : \Sigma^\infty \to \Delta^\infty$ be a morphism. We construct an NDA $\mathcal{A}'$ $= < Q', \Delta, \delta', q_0, F' >$ such that $h(L) = I'(\mathcal{A}')$ as follows. First, for $q \in Q$ and $a \in \Sigma$, define

$$Q_{<q,a>} = \begin{cases} \{ <q, a, 1>, \ldots, <q, a, |h(a)|-1> \} & \text{if } |h(a)| \geqq 2 \\ \phi & \text{if } |h(a)| = 1 \end{cases}$$

Set $S = \cup \{ Q_{<q,a>} \mid q \in Q, a \in \Sigma \}$, and $Q' = Q \cup S \cup \{\#\}$, where $\# \notin Q \cup S$. Define δ' as follows. For $q \in Q$ and $a \in \Sigma$ such that $h(a) \in \Delta$, define δ' by $\delta'(q, h(a)) = \{ \delta(q, b) \mid h(a) = h(b) \}$. For $q \in Q$ and $a \in \Sigma$, let $h(a) = x_1 x_2 \ldots x_n$ $(n \geqq 2)$. Define $\delta'(q, x_1) = \{ <q, b, 1> \mid h(b) = h(a) \}$ and define $\delta'(<q,a,1>, x_2) = \{<q,a,2>\}, \ldots, \delta'(<q,a, |h(a)|-1>, x_n) = \{\delta(q, a)\}$. For any other $<z,a> \in Q' \times \Delta$, define $\delta'(z, a) = \delta'(\#, a) = \{\#\}$. Last $F' = F \cup S$. ::

<u>Theorem 2.3</u>. The class $\mathbb{NL}$ is closed under morphism.

(Proof) Let $L = NL(\mathcal{A})$ for an NDA $\mathcal{A} = <Q, \Sigma, \delta, q_0, F>$, and $h : \Sigma^\infty \to \Delta^\infty$ be a morphism. We construct an NDA $\mathcal{A}' = < Q', \Delta, \delta', q_0, F >$ such that $h(L) = NL(\mathcal{A}')$ as follows. First, for $p, q \in Q$ and $a \in \Sigma$ such that $p \in \delta(q,a)$, define

$$Q_{<q,a,p>} = \begin{cases} \{ <q, a, p, 1>, \ldots, <q, a, p, |h(a)|-1> \} & \\ \phi & \text{if } |h(a)| = 1 \end{cases}$$

Set $S = \cup \{ Q_{<q,a,p>} \mid p \in \delta(q, a) \text{ for some } a \in \Sigma, p, q \in Q \}$, and $Q' = Q \cup S \cup \{\#\}$, where $\# \notin Q \cup S$. Define δ' as follows.

For $q \in Q$ and $a \in \Sigma$ such that $h(a) \in \Delta$, let $\delta'(q, h(a)) = \cup \{ \delta(q, b) \mid h(a) = h(b) \}$. For $q \in Q$ and $a \in \Sigma$, let $h(a) = x_1 x_2 \ldots x_n$ $(n \geqq 2)$ and $p \in \delta(q, a)$. Define $\delta'(q, x_1) = \cup \{ \langle q, b, p, 1 \rangle \mid h(a) = h(b), p \in \delta(q, a) \}$ and define $\delta'(\langle q, a, p, 1 \rangle, x_2) = \{ \langle q, a, p, 2 \rangle \}, \ldots, \delta'(\langle q, a, p, |h(a)|-1 \rangle, x_n) = \delta(q, a)$. For any other $\langle z, a \rangle \in Q' \times \Delta$, define $\delta'(z, a) = \delta'(\#, a) = \{ \# \}$.

Theorem 2.4. The class $\mathbb{E}$ is not closed under morphism.

(Proof) For $\Sigma = \{ a, b \}$, define $h: \Sigma^{\infty} \to \Sigma^{\infty}$ by $h(a) = h(b) = a$. Then $h(X) = a^{\omega} \in \mathbb{E}$ for every ω-language X. ::

Theorem 2.5. The classes $\mathbb{I}$, $\mathbb{L}$ and $\mathbb{L}'$ are not closed under morphism.

(Proof) Since $\mathbb{I} \subsetneq \mathcal{N}\mathbb{I}$, $\mathbb{L} \subsetneq \mathcal{N}\mathbb{L}$, and $\mathbb{L}' \subsetneq \mathcal{N}\subsetneq'$ ([5]), it is obvious from the fact that an ω-language of the class $\mathcal{N}\mathbb{I}$ ($\mathcal{N}\mathbb{L}$, $\mathcal{N}\mathbb{L}'$) is written as an morphic image of an ω-language in Σ ($\mathbb{L}$, $\mathbb{L}'$ resp.). ::

Lemma 2.6. $X = \{ aa, b \}^{*} a^{\omega} \notin \mathcal{N}\mathbb{L}'$.

(Proof) Suppose that $X = NL'(\mathcal{A}_1)$ for an NDA $\mathcal{A}_1 = \langle \Sigma, Q, \delta_1, q_0, F \rangle$. Take b^{ω}, and consider the computation tree of $\mathcal{A}_1$ on b^{ω}, an infinite tree labeled with the elements in Q, defined as follows: The root (of level 0) is labeled with q_0, and if a node v of level n is labeled with a state q then for each q' in $\delta_1(q, w_1[n])$ v has a child labeled with q'. Since $b^{\omega} \notin X$, for every f in F and every path (i.e., run) of the tree, f appears infinitely often on the path. By König's lemma, there exists an integer i such that for every run r in Run($\mathcal{A}_1$, b^ω), F ⊆ {r(0), r(1), ..., r(i)}. Similarly, for an ω-word ab^ω, there exists an integer j such that for every run r' in Run($\mathcal{A}_1$, ab^ω), F ⊆ {r'(0), ..., r'(j)}. Now take an ω-words $w_1 = b^n a^\omega$ with n≧i. Since w_1 ∈ X, there exists a run r" in Run($\mathcal{A}$, w_1) such that F ⊈ Inf(r"). On the other hand, there exist integers k and l such that r'(k) = r"(l) with 1≦k≦i and 1≦l≦j. Last consider $w_2 = ab^l b^{i-k} a^\omega$. Then there exists a run r_1 = r'[l]r"(k)r"(k+1)... ∈ Run($\mathcal{A}_1$, w_2) such that F ⊈ Inf(r_1). This contradicts the fact that w_2 ∉ X. Thus X ∉ $\mathcal{N}\mathbb{L}$'. ::

<u>Theorem 2.7</u>. The class $\mathcal{N}\mathbb{L}$' is not closed under morphism.

(Proof) Let $\mathcal{A}$ be an NDA <{a, b}, {p_0, p_1}, p_0, δ, {p_1}> where for p ∈ Q, δ(p, a) = { q_0 }, δ(p, b) = { p_1 }. Obviously NL'($\mathcal{A}$) = { a, b }$^* a^\omega$. We define a morphism h : { a, b }$^\infty$ → {a, b}$^\infty$ by h(a) = aa and h(b) = b. Then h(NL'($\mathcal{A}$)) = { b, aa }$^* a^\omega$. ::

<u>3. Closure under inverse morphisms</u>

<u>Theorem 3.1 [3, 8]</u>. The classes $\mathbb{R}_\omega$ and $\mathbb{E}$' are closed under inverse ε-free morphisms .

<u>Theorem 3.2</u>. The class $\mathbb{E}$ is closed under inverse ε-free morphisms.

(Proof) Let X = E($\mathcal{A}$) for a DA $\mathcal{A}$ = < Q, Σ, δ, q_0, F>. Without loss of generality, we may assume that for every a ∈ Σ and q ∈ F, δ(q, a) ∈ F. Let h : Δ → Σ be a morphism. We construct an automaton $\mathcal{A}_1 = < Q, \Delta, \delta_1, q_0, F_1 >$ such that $h^{-1}(X) = E(\mathcal{A}_1)$. Define $\delta_1(q, a) = \delta(q, h(a))$ for $q \in Q$ and $a \in \Delta$, and set $F_1 = F$.

<u>Theorem 3.3</u>. The class $\mathbb{I}$ is closed under inverse ε-free morphisms.

(Proof) Let $L = I(\mathcal{A})$ for a DA $\mathcal{A} = < Q, \Sigma, \delta, q_0, F>$, and let $h : \Delta^\infty \to \Sigma^\infty$ be a morphism. We construct an automaton $\mathcal{A}_1 = < Q \cup Q^\sim, \Delta, \delta_1, q_0, Q^\sim >$, where $Q^\sim = \{ q^\sim \mid q \in Q \}$, such that $h^{-1}(L) = I(\mathcal{A}_1)$. The function δ_1 is defined as follows. For $q \in Q$ and $a \in \Delta$ with $h(a) = b_1 \ldots b_n$ $(b_i \in \Sigma)$, $\delta_1(q, a) = \delta_1(q^\sim, a) = \delta(q, h(a))^\sim$ if there exists i $(1 \leqq i \leqq n)$ such that $\delta(q, b_1 \ldots b_i) \in F$, and $\delta_1(q, a) = \delta_1(q^\sim, a) = (\delta(q, h(a))$ otherwise. ::

<u>Theorem 3.4</u>. The class $\mathbb{I}'$ is closed under inverse ε-free morphisms.

(Proof) Let $L = I'(\mathcal{A})$ for a DA $\mathcal{A} = < Q, \Sigma, \delta, q_0, F>$. Let $h : \Delta^\infty \to \Sigma^\infty$ be a morphism. We construct an automaton $\mathcal{A}_1 = < Q \cup Q^\sim, \Delta, \delta_1, q_0, F_1 >$, where $Q^\sim = \{ q^\sim \mid q \in Q \}$, such that $h^{-1}(L) = I'(\mathcal{A}_1)$. The function δ_1 is defined as follows. For $q \in Q$ and $a \in \Delta$ $h(a) = b_1 \ldots b_n$ $(b_i \in \Sigma)$, $\delta_1(q, a) = \delta_1(q^\sim, a) = \delta(q, h(a))$ if there exists i $(1 \leqq i \leqq n)$ such that $\delta(q, b_1 \ldots b_i) \notin F$, and $\delta_1(q, a) = \delta_1(q^\sim, a) = (\delta(q, h(a)))^\sim$ otherwise. Last we set $F_1 = F^\sim$. ::

<u>Lemma 3.5.</u> $X = a(a+b)^* a^\omega \cup b(a+b)^* b^\omega \notin \mathbb{NL}'$.

(Proof) The result can be proved by the simular way as in Lemma 2.6. ::

<u>Theorem 3.6</u>. The classes $\mathbb{L}'$ and $\mathcal{N}\mathbb{L}'$ are not closed under inverse ε-free morphism.

(Proof) Consider the DA $\mathcal{A} \doteq < \{ q_0, q_1, q_2, q_3 \}, \Sigma, \delta, q_0, \{ q_3 \} >$, where $\Sigma = \{ a, b \}$, δ is defined by $\delta(q_0, a) = q_2$, $\delta(q_0, b) = q_1$, $\delta(q_1, a) = q_3$, $\delta(q_1, b) = q_1$, $\delta(q_2, a) = q_2$, $\delta(q_2, b) = q_3$, $\delta(q_3, a) = q_1$, $\delta(q_3, b) = q_2$. Next, define a morphism $h : \Sigma^\infty \to \Sigma^\infty$ by $h(a) = aa$ and $h(b) = bb$. Then $h^{-1}(L'(\mathcal{A}))(= h^{-1}(NL'(\mathcal{A}))) = a(a+b)^* a^\omega \cup b(a+b)^* b$ which is not in $\mathcal{N}\mathbb{L}'$, by the previous Lemma. ::

<u>Theorem 3.7</u>. The classes $\mathbb{L}$ and $\mathcal{N}\mathbb{L}$ are not closed under inverse ε-free morphisms.

(Proof) For $\Sigma = \{ a, b \}$, let $\mathcal{A}$ be a DA $< \{ q_0, q_1, q_2, q_3 \}, \Sigma, \delta, q_0, \{ q_1, q_2 \} >$, where $\delta(q_0, a) = q_1$, $\delta(q_0, b) = q_2$, $\delta(q_1, a) = q_3$, $\delta(q_1, b) = q_2$, $\delta(q_2, a) = q_1$, $\delta(q_2, b) = q_3$, $\delta(q_3, a) = \delta(q_3, b) = q_3$.

Then $L(\mathcal{A}) = (ab)^\omega \cup (ba)^\omega$. Define a morphism $h: \{c, d\}^\infty \to \Sigma^\infty$, by $h(c) = ab$, and $h(d) = ba$. Obviously, $h^{-1}(X) = c^\omega \cup d^\omega$. It is not in $\mathcal{N}\mathbb{L}$ ([5]). ::

<u>Theorem 3.8</u> The class $\mathbb{I}$ is closed under inverse morphism.

(Proof) Let $X = I(\mathcal{A})$ for a DA $\mathcal{A} = < Q, \Sigma, \delta, q_0, F>$, and let $h : \Delta^\infty \to \Sigma^\infty$ be a morphism. We construct an automaton $\mathcal{A}_1 = < Q \cup Q^\sim, \Delta, \delta_1, q_0, Q^\sim >$, where $Q^\sim = \{q^\sim \mid q \in Q\}$, and $\# \notin Q \cup Q^{\sim}$ such that $h^{-1}(X) = I(\mathcal{A}_1)$. The function δ_1 is defined as follows. For $q \in Q$ and $a \in \Delta$ with $h(a) = b_1 \ldots b_n$ ($b_i \in \Sigma$), $\delta_1(q, a) = \delta_1(q^{\sim}, a) = \delta(q, h(a))^{\sim}$ if there exists i ($1 \leqq i \leqq n$) such that $\delta(q, b_1 \ldots b_i) \in F$, and $\delta_1(q, a) = \delta_1(q^{\sim}, a) = (\delta(q, h(a)))$ otherwise. For $q \in Q$ and $a \in \Delta$ with $h(a)=\varepsilon$, $\delta(q, a) = \delta(q^{\sim}, a) = q$.

<u>Theorem 3.9</u> The classes $\mathbb{E}$, $\mathbb{E}'$ and $\mathbb{I}'$ are not closed under inverse morphism.

(Proof) For $\Sigma = \{a, b\}$, let $h: \Sigma^{\infty} \rightarrow \Sigma^{\infty}$ be a morphism defined by $h(a) = a$ and $h(b) = \varepsilon$. Then $h^{-1}(a^{\omega}) = (b^* a)^{\omega} \in \mathbb{I} - \mathbb{E}'$. Define $g: \Sigma^{\infty} \rightarrow \Sigma^{\infty}$ by $g(a) = b$ and $g(b) = \varepsilon$. Thus $g^{-1}(a^* b \Sigma^{\omega}) = (b^* a)^{\omega}$.

<u>Theorem 3.10.</u> The class $\mathbb{R}_{\omega}$ is closed under inverse morphism.

(Proof) Let $X = NI(\mathcal{A})$ for an NDA automaton $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$. Let $h : \Delta^{\infty} \rightarrow \Sigma^{\infty}$ be a morphism. Firsr we construct an automaton $\mathcal{A}_1 = \langle Q \cup Q^{\sim}, \Delta, \delta_1, q_0, F_1 \rangle$, where $Q^{\sim} = \{ q^{\sim} \mid q \in Q \}$. The function δ_1 is defined as follows. For $q \in Q$ and $a \in \Delta$ with $h(a) = \varepsilon$, define $\delta_1(q, a) = \delta_1(q^{\sim}, a) = q$. For $q \in Q$ and $a \in \Delta$ with $h(a) = b_1 \ldots b_n$ ($b_i \in \Sigma$), define $\delta_1(q, a) = \delta_1(q^{\sim}, a) = (\delta(q, h(a)))^{\sim}$ if there exists i ($1 \leqq i \leqq n$) such that $\delta(q, b_1 \ldots b_i) \in F$, and $\delta_1(q, a) = \delta_1(q^{\sim}, a) = \delta(q, h(a))$ otherwise. We set $F_1 = F^{\sim}$. Similarly as in Theorem 3.8, $h^{-1}(NI(\mathcal{A})) = NI(\mathcal{A}_1)$.

4. Duos and generators

Theorem 4.1([4]). (1) $\mathbb{R}_\omega = \mathcal{D}((a^*b)^\omega)$ (2) $\mathbb{E}' = \mathcal{D}(a^\omega)$. ::

Theorem 4.2. $\mathbb{I}' = \mathcal{D}((a+b)^*b^\omega)$

(Proof) Since $(a+b)^*b^\omega \in \mathbb{I}'$, the inclusion $\mathcal{D}((a+b)^*b^\omega) \subseteq \mathbb{I}'$ is obvious by the closure property of $\mathbb{I}'$ under morphisms and inverse ε-free morphisms.

Let $L = I'(\mathcal{A})$ for an NDA $\mathcal{A} = < Q, \Sigma, \delta, q_0, F>$ with $Q = \{q_0, q_1, \ldots, q_n\}$. Let $\Sigma^\sim = \{x^\sim \mid x \in \Sigma\}$ and z a new letter not in $\Sigma \cup \Sigma^\sim$. Define the finite set T by $T = \{ z^i x z^{n-j} \mid x \in \Sigma, q_j \in \delta(q_i, x), q_j \notin F \} \cup \{ z^i x^\sim z^{n-j} \mid x \in \Sigma, q_j \in \delta(q_i, x), q_j \in F \}$. We set $T = \{ t_1, \ldots, t_m \}$ and $Y = \{ y_1, \ldots, y_m \}$. We define the following morphisms:

$h_1 : \{ a, b \}^\infty \to \{ a, b, z \}^\infty$, $h_1(a) = az^n$, $h_1(b) = bz^n$.

$h_2 : (\Sigma \cup \Sigma^\sim \cup \{z\})^\infty \to \{a, b, z\}^\infty$, $h_2(z) = z$, $h_2(x) = a$,

$h_2(x^\sim) = b$ for $x \in \Sigma$.

$h_3 : Y^\infty \to (\Sigma \cup \Sigma^\sim \cup \{z\})^\infty$, $h_3(y_i) = t_i$ for $i = 1, \ldots, m$.

First note that $h_1((a+b)^*b^\omega)) = (az^n+bz^n)^*(bz^n)^\omega$ and thus $h_2^{-1}\cdot h_1((a+b)^*b^\omega)) = (\Sigma z^n+\Sigma^\sim z^n)^*(\Sigma^\sim z^n)^\omega$. Then $L = f(T^\omega \cap (\Sigma z^n+\Sigma^\sim z^n)^*(\Sigma^\sim z^n)^\omega)$, where f is defined by $f(x) = x$, $f(x^\sim) = x$, $f(z) = \varepsilon$ for $x \in \Sigma$. We note that $\cap T^\omega = h_3 \cdot h_3^{-1}$. Thus we get $L = f\cdot h_3\cdot h_3^{-1}\cdot h_2^{-1}\cdot h_1((a+b)^*b^\omega)$. ::

Cororally 4.3. $\mathbb{I}' = \mathcal{D}(a^*b^\omega)$.

(Proof) Define two morphisms $h : \{ c, d, e \}^\infty \to \{ a, b \}^\infty$ and $g : \{ c, d, e \}^\infty \to \{ a, b \}^\infty$ by $h(c) = h(d) = a$, $h(e) = b$ and

$g(c) = a$, $g(d) = g(e) = b$. Then $g \cdot h^{-1}(a^*b^\omega) = (a+b)^*b^\omega$. ::

Cororally 4.4 Every ω-language X in $\mathbb{I}'$ is of the form

$$X = h_5 \cdot h_4^{-1} \cdot h_3 \cdot h_2^{-1} \cdot h_1(a^*b^\omega)$$

where h_1, h_2, h_3, h_4 and h_5 are ε-free morphisms.

Theorem 4.5 [3] (1) Every ω-language X in $\mathbb{E}'$ is of the form $X = h_3^{-1} \cdot h_2 \cdot h_1^{-1}(a^\omega)$ where h_1, h_2 and h_3 are ε-free morphisms.
(2) Every ω-language X in $\mathbb{R}_\omega$ is of the form $X = h_3 \cdot h_2^{-1} \cdot h_1((a^*b)^\omega)$, where h_1, h_2 and h_3 are ε-free morphisms.

Lemma 4.6. $\mathfrak{D}(a^*b(a+b)^\omega) = \mathbb{I}'$.
(Proof) Define the three morphisms $h: \{a, b\}^\infty \to \{a, b, c\}^\infty$ by $h(a) = ac$, $h(b) = bc$, $g: \{a_1, b_1, a_1, b_2\}^\infty \to \{a, b, c\}^\infty$ by $g(a_1) = ac$, $g(b_1) = b$, $g(a) = ca$, $g(b_2) = cb$ and $f: \{a_1, b_1, a_2, b_2\} \to \{a, b\}$ by $f(a_1) = a$, $f(a_2) = f(b_1) = f(b_2) = b$. Then $g^{-1} \cdot h(a^*b(a+b)^\omega = a_1^*b_1(a_2 + b_2)^\omega$. Hence $f \cdot g^{-1} \cdot h(a^*b(a+b)^\omega)$ $= a^*b^\omega$. It is obvious that $\mathfrak{D}(a^*b(a+b)^\omega) = \mathbb{I}'$ by the fact that $a^*b(a+b)^\omega \in \mathbb{I}'$.

Theorem 4.7. $\mathfrak{D}(\mathbb{E}) = \mathfrak{D}(\mathbb{L}') = \mathfrak{D}(\mathcal{N}\mathbb{L}') = \mathbb{I}'$ and $\mathfrak{D}(\mathbb{L}) = \mathfrak{D}(\mathbb{I}) = \mathfrak{D}(\mathcal{N}\mathbb{L}) = \mathbb{R}_\omega$.
(Proof) For an alphabet Σ which contains a and b , $a^*b\Sigma^\omega \in \mathbb{L}' \cap \mathbb{E}$. Define a morphism $h: \Sigma^\infty \to \{a, b\}^\infty$, by $h(a) = a$ and $h(\sigma) = b$ for $\sigma \in \Delta - \{a\}$. Then $h(a^*b\Sigma^\omega) = a^*b(a+b)^\omega$. From Lemma 4.6, $\mathfrak{D}(\mathbb{E}) = \mathfrak{D}(\mathbb{L}') = \mathfrak{D}(\mathcal{N}\mathbb{L}') = \mathbb{I}'$. From the fact that $(a^*b)^\omega \in \mathbb{L}$ ([5]), $\mathfrak{D}(\mathbb{L}) = \mathfrak{D}(\mathcal{N}\mathbb{L}) = \mathfrak{D}(\mathbb{I}) = \mathbb{R}_\omega$. ::

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