数理解析研究所講究録
第 737 巻 1990 年 6-29

Discontinuous Group in a Non-Riemannian Homogeneous Space

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July, 1990

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1. Notations and preliminaries

1.1. proper action

First of all, let us recall the definition of the properness of a continuous map.

DEFINITION 1.1.1. Let $f\colon X \to Y$ be a continuous map between locally compact Hausdorff spaces. f is called *proper* iff one of the following equivalent conditions holds.

(1) f is a closed map, and $f^{-1}(y)$ is compact for any $y \in Y$.

(2) For any topological space Z, $f\colon X \times Z \to Y \times Z$ is a closed map.

(3) $f^{-1}(S)$ is compact for any compact subset S of Y.

If f is a proper map, then it follows easily that a closed subset Z of X is compact iff $f(Z)$ is contained in some compact set of Y.

DEFINITION 1.1.2. The action of a locally compact topological (Hausdorff) group G acting continuously on a locally compact Hausdorff space X is called *proper* iff the map $G \times X \ni (g, x) \mapsto (x, gx) \in X \times X$ is proper. Equivalently, $\{g \in G : f(g, S) \cap S \neq \emptyset\}$ is compact for every compact subset S in X. We call the action is *properly discontinuous* iff G is discrete and acts properly on X.

Suppose that H is a closed subgroup of G. Γ is called *a discontinuous group in* G/H iff Γ is a discrete subgroup of G and Γ acts properly on G/H.

LEMMA 1.1.3. *Let G_i $(i = 1, 2)$ be locally compact groups and $L_i, H_i \subset G_i$ be closed subgroups. Suppose that $f\colon G_1 \to G_2$ is a (continuous) homomorphism such that $f(L_1) \subset L_2$, $f(H_1) \subset H_2$. Assume that $f(L_1)$ is closed in G_2.*

1) *Assume that $L_1 \cap \operatorname{Ker} f$ is compact. If the L_2 action on G_2/H_2 is proper, then the L_1 action on G_1/H_1 is also proper.*

2) *Assume that $f(G_1)H_2 = G_2$, that $G_1 \to G_2/H_2$ is an open map, and that the*

quotients $L_2/f(L_1)$, $f^{-1}(H_2)/H_1$ are compact. If the L_1 action on G_1/H_1 is proper, then the L_2 action on G_2/H_2 is also proper.

REMARK 1.1.4. If G_i are (separable) Lie groups, then it is automatically satisfied from the assumption $f(G_1)H_2 = G_2$ that the map $G_1 \to G_2/H_2$ is open.

PROOF OF LEMMA (1.1.3): 1) Fix any compact subset S of G_1. We have

$$f(L_1 \cap SH_1S^{-1}) \subset L_2 \cap f(S)H_2f(S)^{-1}.$$

If L_2 acts on G_2/H_2 properly, then $f(L_1 \cap SH_1S^{-1})$ is contained in a compact set because $f(S)$ is compact. Then $L_1 \cap SH_1S^{-1}$ is compact, since $f_{|L_1}: L_1 \to L_2$ is a proper map as it is a composition of proper maps: $L_1 \to L_1/L_1 \cap \operatorname{Ker} f \hookrightarrow L_2$. That is, L_1 acts on G_1/H_1 properly.

2) As $f(L_1)$ is a closed and cocompact subgroup of L_2, L_2 acts properly iff $f(L_1)$ acts properly. So we may and do assume $f(L_1) = L_2$. Take a compact set S_1 of G_1 such that $f^{-1}(H_2) = S_1H_1$ and that S_1 contains the unit of G_1. Fix any compact subset S of G_2. We can find a compact subset $\tilde{S}$ of G_1 such that $f(\tilde{S})H_2 \supset S$ as it follows from the assumption that $G_1 \to G_2/H_2$ is an open map and $f(G_1)H_2 = G_2$. Then we have

$$f^{-1}(L_2 \cap SH_2S^{-1}) \subset f^{-1}(L_2) \cap \tilde{S}f^{-1}(H_2)\tilde{S}^{-1}.$$

In particular, ${f_{|L_1}}^{-1}(L_2 \cap SH_2S^{-1})$ is compact if L_1 acts properly on G_1/H_1, because

$${f_{|L_1}}^{-1}(L_2 \cap SH_2S^{-1}) \subset L_1 \cap f^{-1}(L_2) \cap \tilde{S}f^{-1}(H_2)\tilde{S}^{-1} \subset L_1 \cap \tilde{S}S_1H_1{S_1}^{-1}\tilde{S}^{-1}.$$

Under our assumption $f(L_1) = L_2$, we have $L_2 \cap SH_2S^{-1} = f_{|L_1} \circ {f_{|L_1}}^{-1}(L_2 \cap SH_2S^{-1})$ is compact. Thus L_2 acts on G_2/H_2 properly. ∎

1.2. property (CI)

Let H, L be closed subgroups of a locally compact topological group G. If L acts properly on G/H, then any L-orbit is closed with a compact isotropy group. In general, this is not a sufficient condition for the properness of the L-action. Anyway, the second condition about compact isotropy group is easier to check. We call that the triplet (L, G, H) has *property* (CI) iff $L \cap gHg^{-1}$ is compact for any $g \in G$. We call that the triplet (L, G, H) is *proper* iff L acts properly on G/H. Then the following is easily checked from the definition (see [**Bou**] for the first part.)

LEMMA 1.2.1. *With notation as above, the following conditions are equivalent:*

1) (L, G, H) *is proper,*

1)′ (H, G, L) *is proper,*

1)″ $(\operatorname{diag} G, G \times G, H \times L)$ *is proper,*

which imply the following equivalent conditions.

2) (L, G, H) *has property* (CI)

2)′ (H, G, L) *has property* (CI)

2)″ $(\operatorname{diag} G, G \times G, H \times L)$ *has property* (CI)

As we mentioned above, it is easier to check property (CI) than properness. So we are interested in how property (CI) approximates properness.

EXAMPLE 1.2.2.

1) Suppose that G is a linear reductive Lie group, and that H, L are closed subgroups reductive in G (see §1.3 for definition). Then property (CI) $\Leftrightarrow$ properness. This is a

restatement of one of our main results in [**Ko**], Theorem 4.1.

2) Suppose that G is a linear reductive noncompact Lie group. Let $G = KAN$ be an Iwasawa decomposition and let $H := A$, $L := N$. Then property (CI) is always satisfied for (L, G, H), while L never acts properly on G/H.

3) If L is normal in G and if HL is closed, then property (CI) $\Leftrightarrow$ properness.

4) Suppose that $G = GL(2, \mathbb{R}) \ltimes \mathbb{R}^2$, $H = GL(2, \mathbb{R})$. Then for any connected closed Lie subgroup L of G, property (CI) $\Leftrightarrow$ properness.

The proof of (2), (3) is easy. As for (4), we shall classify the maximal connected Lie groups L of G such that (L, G, H) has property (CI) in the proof in §2.2, which we also see is in fact proper.

1.3. notations for reductive groups

In this subsection we set up notation.

Let G be a real linear reductive Lie group, with real Lie algebra $\mathfrak{g}$. Given a Cartan involution θ of G, we always write a Cartan decomposition of its Lie algebra as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Fix a maximally abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. $\mathfrak{a}$ is called a maximally split abelian subspace for G. We write $W(\mathfrak{g}, \mathfrak{a})$ for the Weyl group associated to the root system of $\Sigma(\mathfrak{g}, \mathfrak{a})$, $\mathbb{R}\text{-rank}\, G := \dim \mathfrak{a}$ $(\leq \operatorname{rank} G \geq)$ $c\text{-rank}\, G := \operatorname{rank} K$, and $\mathrm{d}(G) := \dim G/K = \dim \mathfrak{p}$. Let H be a closed subgroup in G. If there exists a Cartan involution of G which stables H, then H is called *reductive in* G and G/H is called *a homogeneous space of reductive type*. In this case, H is of finite connected components, H has a Cartan decomposition $H = (H \cap K)\exp(\mathfrak{h} \cap \mathfrak{p})$, and $\mathfrak{h}$ is reductive in $\mathfrak{g}$, namely, the adjoint representation $\mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ is completely reducible. Let $\mathfrak{a}_H$ be a maximally split abelian subspace for H.

Then there exists an element g of G such that $\mathrm{Ad}(g)\mathfrak{a}_H \subset \mathfrak{a}$. Put $\mathfrak{a}(H) := \mathrm{Ad}(g)\mathfrak{a}_H$, which is uniquely defined up to conjugacy of $W(\mathfrak{g}, \mathfrak{a})$.

REMARK 1.3.1. Definition-Lemma (2.6) in **[Ko]** is not accurate if H is not an algebraic group defined over $\mathbb{R}$ (cf. [M]). Our definition here is equivalent to (2.6.1), and implies (2.6.2) there. Any statement there is valid for a homogeneous space of reductive type in this sense.

We will use the standard notation $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Here $\mathbb{N}$ means the set of non-negative integers and $\mathbb{H}$ means the $\mathbb{R}$-algebra of quarternionic numbers.

2. Homogeneous spaces of semidirect product groups

2.1. semidirect product

Proposition 2.1.1 Let G be a Lie group and H be a closed subgroup. Assume that $\mathfrak{h}$ contains a maximal semisimple algebra of $\mathfrak{g}$. Then any connected closed subgroup L such that (L, G, H) has property (CI) is amenable.

PROOF: Let $\mathfrak{l} = \mathfrak{l}_s + \mathfrak{l}_n$ be a Levi decomposition of $\mathfrak{l}$, where $\mathfrak{l}_s$ is a maximal semisimple algebra and $\mathfrak{l}_n$ is the radical. It follows from the assumption that there exists $g \in G$ such that $\mathfrak{l}_s \subset \mathrm{Ad}(g)\mathfrak{h}$. Thus, $L \cap gHg^{-1} \supset L_s$, where L_s is a connected semisimple

Lie subgroup with Lie algebra $\mathfrak{l}_s$. Therefore L_s must be compact. L is thus a compact extension of a solvable group, namely, an amenable group. ▮

2.2. affine transformation group of $\mathbf{R}^2$

Let $G = GL(2, \mathbf{R}) \ltimes \mathbf{R}^2$, the affine transformation group of $\mathbf{R}^2$. The multiplicative structure is given by $(g_1, v_1) \cdot (g_2, v_2) := (g_1 g_2, g_1 v_2 + v_1)$, where $g_i \in GL(2, \mathbf{R})$, $v_i \in \mathbf{R}^2$. The Lie algebra $\mathfrak{g}$ is identified with $M(3, 2; \mathbf{R}) = \{(A, u) : A \in \mathfrak{gl}(2, \mathbf{R}),\ u \in \mathbf{R}^2\}$ equipped with $[(A_1, u_1), (A_2, u_2)] = ([A_1, A_2], A_1 u_2 - A_2 u_1)$. The adjoint action is given by $\mathrm{Ad}((g, v))(A, u) = (gAg^{-1}, gu - gAg^{-1}v)$. Let $H = GL(2, \mathbf{R})$, the isotropy subgroup of G at $0 \in \mathbf{R}^2$. Here is a classification of maximal connected Lie groups acting properly on $G/H \simeq \mathbf{R}^2$.

PROPOSITION 2.2.1. *Up to conjugacy the maximal connected Lie subgroups of G acting properly on G/H are of the following forms;*

$$L_1 = \left\{ \begin{pmatrix} e^b & 0 & a \\ 0 & 1 & b \end{pmatrix} : a, b \in \mathbf{R} \right\},$$

$$L_2 = \left\{ \begin{pmatrix} 1 & b & a \\ 0 & 1 & b \end{pmatrix} : a, b \in \mathbf{R} \right\},$$

$$L_3 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \end{pmatrix} : a, b, \theta \in \mathbf{R} \right\}.$$

It can be checked directly that L_i acts properly on G/H ($i = 1, 2, 3$). Conversely, if a connected group L acts properly on G/H, then (L, G, H) has property (CI). We shall classify L such that (L, G, H) has property (CI) in the following way. First, L is a compact extension of a solvable group from Proposition (2.1.1). In our case, a maximal compact sub group of G is of one dimension, and thus L itself is a solvable Lie group.

So we can take a sequence $0 = \mathfrak{l}^{(0)} \triangleleft \mathfrak{l}^{(1)} \triangleleft \cdots \triangleleft \mathfrak{l}^{(n)} = \mathfrak{l}$ such that $\mathfrak{l}^{(i)}$ is a codimension one ideal in $\mathfrak{l}^{(i+1)}$. (It is easy to see that $n \leq 3$.) Now checking property (CI) is reduced to the calculation of the normalizer $N_{\mathfrak{g}}(\mathfrak{l}^{(i)})$ and to the case of $\dim L = 1$ (Lemma (2.2.3)). The rest of this section is devoted to complete the proof of Proposition (2.2.1) by this procedure.

LEMMA 2.2.2. *A complete representative of the adjoint orbit in* $\mathfrak{g}$ *is given by*

$$X(a,b) := \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \end{pmatrix} \quad (a, b \in \mathbf{R},\ a \leq b), \quad W(a) := \begin{pmatrix} 0 & 0 & 1 \\ 0 & a & 0 \end{pmatrix} \quad (a \in \mathbf{R}),$$

$$Y(a) := \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \end{pmatrix} \quad (a \in \mathbf{R}), \qquad V := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$Z(a,b) := \begin{pmatrix} a & -b & 0 \\ b & a & 0 \end{pmatrix} \quad (a, b \in \mathbf{R},\ b > 0).$$

LEMMA 2.2.3. *Up to conjugacy, the one dimensional connected Lie subgroups of* G *which act properly on* G/H *have one of the following Lie algebras:* $\mathbf{R}Z(0,1)$, $\mathbf{R}W(1)$, $\mathbf{R}W(0), \mathbf{R}V$.

PROOF: We notice that if $a \neq 0$ then there exists $g \in G$ such that $\mathrm{Ad}(g)\mathbf{R}W(a) = \mathbf{R}W(1)$. So the necessity is shown by checking the property (CI). We have already seen the sufficiency before. ∎

The proof of the following two lemmas is straightforward and so omitted.

LEMMA 2.2.4. *The normalizers of the Lie algebras in Lemma (2.2.3) are given by,*

$$N_{\mathfrak{g}}(\mathbf{R}Z(0,1)) = \mathbf{R}Z(0,1) + \mathbf{R}Z(1,0),$$

$$N_{\mathfrak{g}}(\mathbf{R}W(0)) \quad = \{X \in M(3,2;\mathbf{R}) : X_{21} = 0\},$$

$$N_{\mathfrak{g}}(\mathbf{R}W(1)) \quad = \mathbf{R}W(1) + \mathbf{R}W(0),$$

$$N_{\mathfrak{g}}(\mathbf{R}V) \qquad = \mathbf{R}V + \mathbf{R}X(2,1) + \mathbf{R}W(0).$$

Set $W'(a) := \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $(a \in \mathbf{R})$, $V' := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, which are conjugate to $W(a)$, V respectively. Put $P := N_G(\mathbf{R}W(0))$, $Q := N_G(\mathbf{R}V) \subset G$.

LEMMA 2.2.5.

(1) $\mathrm{Ad}(G)Z(a,b) \cap \mathfrak{p} = \emptyset$ *if* $b \neq 0$.

(2) $\mathrm{Ad}(G)W(a) \cap \mathfrak{p} = \mathrm{Ad}(P)W(a) \coprod \mathrm{Ad}(P)W'(a) \quad (a \in \mathbf{R})$.

(3) $\mathrm{Ad}(G)V \cap \mathfrak{p} = \mathrm{Ad}(P)V$.

(4) $\mathrm{Ad}(G)Z(a,b) \cap \mathfrak{q} = \emptyset$ *if* $b \neq 0$.

(5) $\mathrm{Ad}(G)W(a) \cap \mathfrak{q} = \begin{cases} \emptyset & \text{if } a \neq 0 \\ \mathbf{R}^{\times}W(0) = \mathrm{Ad}(Q)W(0) & \text{if } a = 0 \end{cases}.$

(6) $\mathrm{Ad}(G)V \cap \mathfrak{q} = \coprod_{c \in \mathbf{R}} \mathrm{Ad}(Q)(V + cW(0))$.

LEMMA 2.2.6. *Up to conjugacy the two dimensional connected Lie subgroups L of G which act properly on G/H are of the following Lie algebras:*

$\mathbf{R}W'(0) + \mathbf{R}W(0)$, $\mathbf{R}W'(1) + \mathbf{R}W(0)$, $\mathbf{R}V + \mathbf{R}W(0)$.

PROOF: We have seen already that the corresponding Lie subgroups in Lemma (2.2.3) act properly on G/H. Let us verify the necessity part by the property (CI). As $\mathfrak{l}$ is a solvable Lie algebra, we can assume that one of the Lie algebras in Lemma (2.2.3) is an ideal of $\mathfrak{l}$. First consider the case where $\mathbf{R}W(0) \triangleleft \mathfrak{l}$. Then $\mathfrak{l} \subset N_{\mathfrak{g}}(\mathbf{R}W(0)) = \mathfrak{p}$. Up to conjugacy by P, we may assume that a complimentary subspace of $\mathbf{R}W(0)$ in $\mathfrak{l}$ is one of $\mathbf{R}W(0)$, $\mathbf{R}W(1), \mathbf{R}W'(0)$, $\mathbf{R}W'(1)$, $\mathbf{R}V$ from Lemma (2.2.5). Then $\mathbf{R}W(0)$ is excluded because of linear dependency. $\mathbf{R}W(1)$ is also excluded because $\mathbf{R}W(0)+\mathbf{R}W(1)$ contains a subspace $\mathbf{R}(W(1) - W(0))$, whose corresponding connected Lie subgroup cannot act properly on G/H. The remaining is properly discontinuous cases. Similarly, we can treat the cases where $\mathbf{R}Z(0,1) \triangleleft \mathfrak{l}$, $\mathbf{R}W(1) \triangleleft \mathfrak{l}$, $\mathbf{R}V \triangleleft \mathfrak{l}$, yielding Lemma. ∎

The final step is done similarly by using the following lemma.

LEMMA 2.2.7. *The normalizers of the Lie algebras in Lemma (2.2.6) are given by,*

$$\begin{aligned} N_{\mathfrak{g}}(\mathbf{R}W'(0) + \mathbf{R}W(0)) &= \mathfrak{g}, \\ N_{\mathfrak{g}}(\mathbf{R}W'(1) + \mathbf{R}W(0)) &= \mathbf{R}X(1,0) + \mathbf{R}W'(1) + \mathbf{R}W(0), \\ N_{\mathfrak{g}}(\mathbf{R}V + \mathbf{R}W(0)) &= \mathbf{R}X(2,1) + \mathbf{R}Y(0) + \mathbf{R}V + \mathbf{R}W(0). \end{aligned}$$

3. Homogeneous spaces of solvable groups

First we recall a nice topological property of a subgoup of a solvable Lie group due to Chevalley.

FACT 3.1, [**Ch**]. *Let G be a 1-connected (real) solvable group and H be a connected subgroup of G. Then H is closed and 1-connected.*

Our main theorem in this section is,

THEOREM 3.2. *Let G be a connected (real) solvable group and H be a closed proper subgroup of G. If the fundamental group $\pi_1(G/H)$ is finite, then there exists a discontinuous group in G/H which is isomorphic to $\mathbb{Z}$.*

This result should be in sharp contrast to the case of homogeneous spaces of reductive type, which is a phenomenon first observed in [**C-M**] and is settled in general in [**Ko**].

FACT 3.3, [**C-M; Wo1; Wo2; Ku; Ko**]. *Let G/H be a homogeneous space of reductive type. Then the followings are equivalent:*

(1) *Any discontinuous group in G/H is finite.*
(2) $\mathbb{R}\text{-rank}\, G = \mathbb{R}\text{-rank}\, H$.

A stupid observation is when G is solvable and reductive, namely, G is isomorphic to $\mathbb{R}^m \times \mathbb{T}^n$. Suppose that the first Betti number of H is n'. Then obviously,

$$|\pi_1(G/H)| < \infty \Longleftrightarrow n = n' \Longleftrightarrow G = H \text{ or } \mathbb{R}\text{-rank}\, G > \mathbb{R}\text{-rank}\, H.$$

This means a compatibility of Theorem (3.2) and Fact (3.3).

Thanks to Lemma (1.1.3)(2) with G_1 a universal covering group of $G_2 := G$ and with H_1 a connected subgroup of G_1 having the same Lie algebra of $\mathfrak{h}_2 := \mathfrak{h}$, Theorem (3.2) is reduced to the following Theorem (3.2)$'$.

THEOREM 3.2$'$. *Let G be a 1-connected (real) solvable group and H be a connected proper subgroup of G. Then there exists a discontinuous group in G/H which is isomorphic to $\mathbb{Z}$.*

PROOF: We proceed by the induction on the dimension of G. Theorem $(3.2)'$ is clear when $\dim G = 1$, namely, when $G \simeq \mathbf{R} \supset H \simeq \{0\}$. Suppose that $\dim G \geq 2$. Then there exists a connected normal subgroup N of G with $0 < \dim N < \dim G$. We will divide into two cases according as $HN \subsetneqq G$ or $HN = G$.

I) Assume that $HN \subsetneqq G$. A subgroup HN is connected and so closed. So $\overline{H} := H/H \cap N = HN/N$ is a proper closed subgroup of $\overline{G} := G/N$. We write the canonical projection $\pi \colon G \to \overline{G} = G/N$. From the inductive assumption, we can find a discrete group $\overline{\Gamma}$ of $\overline{G}$ such that $\overline{\Gamma}$ is isomorphic to $\mathbb{Z}$ and acts on $\overline{G}/\overline{H}$ properly. Fix an element $\gamma \in G$ such that $\pi(\gamma)$ is a generator of $\overline{\Gamma}$. Put $\Gamma := \langle \gamma \rangle$. We have $\pi(\Gamma) = \overline{\Gamma}$, and therefore $\Gamma \simeq \mathbb{Z}$ and $\Gamma \cap N = \{e\}$. On the other hand, $\overline{\Gamma}$ is discrete and so does Γ. Applying Lemma (1.1.3)(1), we have now shown that Γ acts on G/H properly discontinuously.

II) Assume that $HN = G$. We have $G/H \simeq N/N \cap H$ and $N \cap H \subsetneqq N$. Since $\pi_1(N/N \cap H) = \pi_1(G/H) = \{e\}$, $N \cap H$ is connected. Thus $(N, N \cap H)$ satisfies the assumption of Theorem $(3.2)'$ and $\dim N < \dim G$. Therefore we can find a discrete group $\Gamma \simeq \mathbb{Z}$ of N which acts on $N/N \cap H$ from the inductive assumption. Clearly, Γ is a subgroup of G acting properly discontinuously on G/H. ▮

4. R-rank one semisimple group manifolds

Throughout this section, we assume that G is a connected real reductive linear Lie group. See §1.3 for notations. We shall find some property of a discontinuous group in a group manifold $G \times G / \operatorname{diag} G$ when $\mathbf{R}$-rank $G = 1$.

LEMMA 4.1.1. *If* $\mathbf{R}$-rank $G = 1$ *and* $x \in G$ *is a semisimple and non-elliptic element, then* $Z_G(x)$ *is a direct product of a compact group and* $\mathbf{R}$.

PROOF: Choose a Cartan subgroup J of G containing x and a Cartan involution θ such that $\theta J = J$. Put $L := Z_G(x)$, then we have $\theta \mathfrak{l} = \mathfrak{l}$ (see [**War**] Proposition 1.4.3.2). As $\mathfrak{l}$ is of maximal rank reductive Lie subalgebra of $\mathfrak{g}$, we have $N_G(\mathfrak{l}) \supset L \supset L_0$ have the same Lie algebra ([**War**] Proposition 1.4.2.4). Since $\theta N_G(\mathfrak{l}) = N_G(\mathfrak{l})$, we have $\theta L = L$. So we can write the center of L as $C = (C \cap K) \exp(\mathfrak{c} \cap \mathfrak{p})$. As L is a reductive linear Lie group with finitely many connected components, if follows from $\langle x \rangle \simeq \mathbf{Z} \subset C$ that $\dim \mathfrak{c} \cap \mathfrak{p} \geq 1$. Then $\mathfrak{c} \cap \mathfrak{p} = \mathfrak{p}$ because $1 = \mathbf{R}\text{-rank}\, G \geq \mathbf{R}\text{-rank}\, L = \mathbf{R}\text{-rank}[L, L] + \dim \mathfrak{c} \cap \mathfrak{p}$. Hence $L = (L \cap K) \exp(\mathfrak{c} \cap \mathfrak{p})$. ∎

LEMMA 4.1.2. *If* $\mathbf{R}$-rank $G = 1$ *and* Γ *is an infinite discrete subgroup of* G*, then there exists a compact set* S *of* G *such that* $S \Gamma S^{-1} = G$.

PROOF: An infinite discrete subgroup Γ in a linear Lie group contains necessarily an element of infinite order because Γ has a torsion-free subgroup Γ' such that $[\Gamma : \Gamma'] < \infty$ ([**Se**]). Thus it suffices to show Lemma (4.1.2) when Γ is isomorphic to $\mathbf{Z}$. Let γ be a generator of Γ and $\gamma = \gamma_s \gamma_u$ be a Jordan decomposition (see [**War**] Proposition 1.4.3.3). We divide into two cases according as $\langle \gamma_s \rangle$ is discrete in G or not.

I) Assume that $\langle \gamma_s \rangle$ is discrete in G. It follows from Lemma (4.1.1) that $\gamma_u \in Z_G(\gamma_s)$ is the identity. Thus $\gamma = \gamma_s$ is contained in a maximally split Cartan subgroup J. Choose a Cartan involution θ which stables J and we write $J = TA$ as usual. We can write $\gamma = t \exp(Y)$ where $t \in T$, $Y \in \mathfrak{a}$. Define a compact subset of G by $S :=$

$K\{\exp sY : 0 \leq s \leq 1\}$. Then $S\langle\gamma\rangle S^{-1} \supset KAK = G$.

II) Assume that $\langle\gamma_s\rangle$ is not discrete in G. Then $\gamma_u \neq 1$ since $\langle\gamma\rangle = \{{\gamma_s}^n{\gamma_u}^n : n \in \mathbb{Z}\}$ is discrete in G. By the theorem of Jacobson-Morozov, there is a homomorphism $\psi\colon SL(2,\mathbf{R}) \to G$ such that $\psi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \gamma_u$. There is a Cartan involution θ of G such that $\theta\psi(SL(2,\mathbf{R})) = \psi(SL(2,\mathbf{R}))$ (see **[He]**, p.277). In particular, $A := \psi\left(\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0\right\}\right)$ is a maximally split abelian subgroup of G. Define a compact subset of G by $S := K\psi\left(\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : 0 \leq x \leq 1\right\}\right)\overline{\langle\gamma_s\rangle}$. Then

$$S\langle\gamma\rangle S^{-1} \supset K\psi\left(\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{R}\right\}\right)K \supset K\psi\left(SL(2,\mathbf{R})\right)K \supset KAK = G.$$

∎

LEMMA 4.1.3. *Let G be a connected reductive Lie group. Then the following conditions are equivalent.*

(1) $\mathbf{R}$-rank $G \geq 2$

(2) *There exists infinite discrete subgroups Γ_i of G $(i = 1,\ 2)$ such that $\Gamma := \Gamma_1 \times \Gamma_2$ acts properly discontinuously on a group manifold $G \times G/\operatorname{diag} G$.*

PROOF: We may restrict ourselves to the case where $\mathbf{R}$-rank $G \geq 1$.

Suppose that $\mathbf{R}$-rank $G \geq 2$. We find abelian subspaces $\mathfrak{a}_1,\ \mathfrak{a}_2 \subset \mathfrak{a}$ such that $\dim \mathfrak{a}_i \geq 1$ and that $W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_1 \cap \mathfrak{a}_2 = \{0\}$. Put $A_i := \exp \mathfrak{a}_i$, then A_1 acts properly on G/A_2. Take any lattices Γ_i in abelian Lie groups A_i $(i = 1, 2)$. Then the discrete group $\Gamma_1 \times \Gamma_2$ acts properly discontinuously on $G \times G/\operatorname{diag} G$ (see Lemma (1.2.1)).

Suppose that $\mathbf{R}$-rank $G = 1$. Let Γ_i $(i = 1,\ 2)$ be both infinite discrete subgroups of G. Then there exists a compact set S of G such that $S\Gamma_i S^{-1} = G$ by Lemma (4.1.2).

In particular, $(S \times S)(\Gamma_1 \times \Gamma_2)(S^{-1} \times S^{-1}) = G \times G$, which implies that any subgroup H of $G \times G$ acting properly on $G \times G/\Gamma_1 \times \Gamma_2$ must be compact. Thus, $\Gamma_1 \times \Gamma_2$ cannot act properly discontinuously on $G \times G/\operatorname{diag} G$. ▮

THEOREM 4.1.4. *Let G be a connected noncompact reductive Lie group. Then the following conditions are equivalent.*

(1) $\mathbf{R}$-rank $G = 1$

(2) *Any torsionless discontinuous group Γ in $G \times G/\operatorname{diag} G$ is of the following form up to switch of factor: $\Gamma = \{(\gamma, \rho(\gamma)) : \gamma \in \Phi\}$ with a subgroup $\Phi \subset G$ and with a homomorphism $\rho\colon \Phi \to G$.*

PROOF: 2) $\Rightarrow$ 1) If $\mathbf{R}$-rank $G \geq 2$, then there exists a discrete group $\Gamma_i \simeq \mathbb{Z}^{n_i}$ $(n_i \geq 1)$ of G such that $\Gamma_1 \times \Gamma_2$ acts properly discontinuously on $G \times G/\operatorname{diag} G$ as we saw it in the previous lemma.

1) $\Rightarrow$ 2) Suppose that Γ is a discontinuous group in $G \times G/\operatorname{diag} G$. Let $p_j\colon G \times G \to G$ $(j = 1, 2)$ be natural projections to the j-th factor. Let $\Gamma_j := \operatorname{Ker} p_j \cap \Gamma$ for $j = 1, 2$. Then $\Gamma_1 \times \Gamma_2$ is regarded as a subgroup of $\Gamma \subset G \times G$, and so is also a discontinuous group in $G \times G/\operatorname{diag} G$. It follows from the previous Lemma that at least one of Γ_j must be finite if $\mathbf{R}$-rank $G = 1$. We can assume Γ_1 is a finite group after changing factor if necessary. As Γ is torsion-free, a finite subgroup Γ_1 must be trivial, namely, $p_{1|\Gamma}\colon \Gamma \to G$ is injective. Now Γ is of the desired form if we define $\Phi := p_1(\Gamma)$ and $\rho := p_2 \circ p_{1|\Gamma}^{-1}$. ▮

REMARK. R.Kulkarni and F.Raymond first proved (1) $\Rightarrow$ (2) when $G = SL(2, \mathbf{R})$ (see Theorem 5.2 and Introduction in [**Ku-R**]). They also show that Ψ can be chosen to be discrete. The proof there depends on the fact that no discontinuous group in $G \times$

$G/\operatorname{diag} G$ contains a subgroup $\simeq \mathbb{Z}^2$ if $G = SL(2,\mathbb{R})$. However, this is not always true even if G is of $\mathbb{R}$-rank one. For example, there exists a discontinuous group $\simeq \mathbb{Z}^{n-1}$ in $G \times G/\operatorname{diag} G$ if $G = SO(n,1)$.

5. A necessary condition for the existence of a uniform lattice

5.1. theorem

A homogeneous space of reductive type G/H does not always admit a uniform lattice. There are known two necessary conditions for the existence of a uniform lattice. One is the requirement that there should exist a discontinuous group $\simeq \mathbb{Z}$ in G/H (see Fact(3.3)), and the other is a requirement from Euler characteristic ([**Ko**] Proposition (4.10), see also [**Ku**] Corollary 2.10, [**Ko-O**] Corollary 5 for partial results):

FACT 5.1.1. *For the existence of a uniform lattice, (G, H) must satisfy that*

1) $\mathbb{R}\text{-rank}\, G > \mathbb{R}\text{-rank}\, H$ *unless G/H itself is compact.*

2) *If* $\operatorname{rank} G = \operatorname{rank} H$ *then* $c\text{-rank}\, G = c\text{-rank}\, H$.

By a comparison with various reductive subgroups in G, we can exclude the possibility of the existence of uniform lattice in some of homogeneous spaces of reductive type. The following simple theorem is based on this idea.

THEOREM 5.1.2. *Let G/H be a homogeneous space of reductive type. If there exists a closed subgroup G' reductive in G such that*

$$(5.1.3)(a) \qquad \mathfrak{a}(G') \subset W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}(H)$$

$$(5.1.3)(b) \qquad \mathrm{d}(G') > \mathrm{d}(H)$$

then G/H does not admit a uniform lattice (see §1.3 for notations).

PROOF: Suppose that there were $\Gamma \subset G$, a uniform lattice in G/H. Then Γ is virtually torsionless and the cohomological dimension $\mathrm{cd}_{\mathbb{R}}(\Gamma) = \mathrm{d}(G) - \mathrm{d}(H)$ from Corollary 5.5 (1) in [**Ko**]. On the other hand, the condition (5.1.3)(a) implies that Γ acts on G/G' properly discontinuously. Using Corollary 5.5 in [**Ko**] again, we have $\mathrm{cd}_{\mathbb{R}}(\Gamma) \leq \mathrm{d}(G) - \mathrm{d}(G')$. Thus $\mathrm{d}(G') \leq \mathrm{d}(H)$, which contradicts (5.1.3)(b). ∎

REMARK 5.1.4. One of the simplest applications is a comparison of G/H with G/G by taking $G' = G$, yielding Fact (5.1.1)(1). Indeed, if $\mathbb{R}\text{-rank}\, G = \mathbb{R}\text{-rank}\, H$, then Theorem (5.1.2) implies

$$G/H \text{ has a uniform lattice} \iff \mathrm{d}(G) = \mathrm{d}(H) \iff G/H \text{ is compact}.$$

Here, the second equivalence is derived immediately from a fiber bundle structure $G/H \simeq K/H \cap K \underset{H \cap K}{\times} \mathfrak{p}/\mathfrak{h} \cap \mathfrak{p}$.

The proof of Theorem (5.1.2) is almost obvious as we saw above. Throughout the rest of this section we will clarify its typical applications and limitations.

5.2. example

EXAMPLE 5.2.1. Let $G/H = U(i+j, k+l; \mathsf{F})/U(i, k; \mathsf{F}) \times U(j, l; \mathsf{F})$, where $\mathsf{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $i \le j, k, l$. Here, we use a notation: $U(p, q; \mathbb{R}) = SO(p, q)$, $U(p, q; \mathbb{C}) = U(p, q)$, and $U(p, q; \mathbb{H}) = Sp(p, q)$ (of rank $p+q$). If G/H admits a uniform lattice, then G/H is compact ($i = j = 0$ or $i = k = 0$), or H is compact ($i = l = 0$), or $0 = i < l \le j - k$.

PROOF: To see the condition $0 = i < l \le j - k$, it suffices to apply Theorem (5.1.2) with $G' = G_i$, where $G_1 := U(i+t, k+l; \mathsf{F})$, $G_2 = U(i+j, i+t; \mathsf{F})$, and $t := \min(j, l)$. ∎

REMARK 5.2.2. Assume moreover that $\mathsf{F} = \mathbb{R}$ in the above Example. Then it is also necessary that $jkl \equiv 0 \mod 2$ from Fact (5.1.1)(2). Conversely, it is known that if $i = l = 0$ or if $(i, j, k, l) = (0, 2n, 1, 1), (0, 4n, 1, 3), (0, 4, 2, 1)$, then there exists a uniform lattice in G/H.

5.3. semisimple orbit

Let us apply Theorem (5.1.2) to a semisimple orbit of the adjoint action. First we fix notations. Let G be a connected real linear reductive Lie group and X be an element of its Lie algebra $\mathfrak{g}$. $G \cdot X \simeq G/Z_G(X)$ is called an (adjoint) orbit in $\mathfrak{g}$, where $G \cdot X := \{\mathrm{Ad}(g)X : g \in G\}$, $Z_G(X) := \{g \in G : \mathrm{Ad}(g)X = X\}$.

COROLLARY 5.3.1. *In the above setting, suppose that X is a semisimple element. If $G \cdot X \simeq G/Z_G(X)$ admits a uniform lattice, then there is an elliptic element $X_1 \in \mathfrak{g}$ such that $Z_G(X) = Z_G(X_1)$. In particular, the orbit $G \cdot X$ carries a G-invariant complex structure.*

REMARK 5.3.2. We should note that G itself never carries a complex Lie group structure if $G \cdot X \simeq G/Z_G(X)$ admits a uniform lattice with a nonzero semisimple element

$X \in \mathfrak{g}$. This follows from the fact that $\mathbf{R}\text{-rank}\, Z_G(X) = \mathbf{R}\text{-rank}\, G$ if G is a complex reductive Lie group.

PROOF OF COROLLARY (5.3.1): There exists a Cartan involution θ which stables $Z_G(X)$. Since $\operatorname{rank} Z_G(X) = \operatorname{rank} G$ (**[War]** Proposition 1.4.3.2) and since $G/Z_G(X)$ admits a uniform lattice, we have $c\text{-rank}\, Z_G(X) = c\text{-rank}\, G$ from Fact (5.1.1)(2). Therefore $Z_{\mathfrak{g}}(X)$ contains a fundamental Cartan subalgebra $\mathfrak{j}^c$ of $\mathfrak{g}$. We may and do assume $\mathfrak{j}^c$ is θ-stable and we write $X = X_1 + X_2$ corresponding to the direct sum decomposition $\mathfrak{j}^c = \mathfrak{t}^c + \mathfrak{a}^c := \mathfrak{j}^c \cap \mathfrak{k} + \mathfrak{j}^c \cap \mathfrak{p}$. Then the first statement of Corollary follows from Theorem (5.1.2) with $H = Z_G(X) \subset G' = Z_G(X_1)$ combined with the following Claim (5.3.3) The last statement is directly from a (generalized) Borel embedding. ∎

CLAIM 5.3.3. *With notation as above, we have*

1) $Z_G(X) = Z_G(X_1) \cap Z_G(X_2)$,

2) $\mathbf{R}\text{-rank}\, Z_G(X) = \mathbf{R}\text{-rank}\, Z_G(X_1)$,

3) *either* $Z_G(X) = Z_G(X_1)$ *or* $\mathrm{d}(Z_G(X)) < \mathrm{d}(Z_G(X_1))$.

PROOF OF CLAIM (5.3.3): The first claim is a direct consequence of the equation $Z_G(X) = \theta(Z_G(X)) = Z_G(\theta X)$. Take a maximally split abelian subspace $\mathfrak{a}_1$ of $Z_{\mathfrak{g}}(X_1)$ such that $\mathfrak{a}_1$ contains X_2. This is possible because $X_2 \in \mathfrak{p} \cap Z_{\mathfrak{g}}(X_1)$. Then $\mathfrak{a}_1 \subset Z_{\mathfrak{g}}(X_1) \cap Z_{\mathfrak{g}}(X_2) = Z_{\mathfrak{g}}(X)$. This means that $\mathfrak{a}_1$ is also a maximally split abelian subspace of $Z_{\mathfrak{g}}(X)$, whence the second part.

Let us prove the third part. Suppose $Z_G(X) \subsetneqq Z_G(X_1)$. As the centralizer of an elliptic element is necessarily connected, it follows that $Z_{\mathfrak{g}}(X) \subsetneqq Z_{\mathfrak{g}}(X_1)$. Noting $\mathfrak{j}^c \subset Z_{\mathfrak{g}}(X) \subsetneqq Z_{\mathfrak{g}}(X_1)$, we find an $\alpha \in \Delta(Z_{\mathfrak{g}}(X_1), \mathfrak{j}^c) \setminus \Delta(Z_{\mathfrak{g}}(X), \mathfrak{j}^c)$. If we write

$\alpha = \alpha_1 + \alpha_2$ corresponding to the direct sum decomposition $\mathfrak{j}^{c*} = \mathfrak{t}^{c*} + \mathfrak{a}^{c*}$, we have $\alpha_1(X_1) = 0$, $\alpha(X) = \alpha_1(X_1) + \alpha_2(X_2) \neq 0$. Fix a nonzero element $Y \in \mathfrak{g}(\mathfrak{j}^c; \alpha)$ and set $Z := Y - \theta Y \in \mathfrak{p}$. Since $\alpha \neq \theta\alpha$, Y and θY are linearly independent. Now we have $[X_1, Z] = \alpha_1(X_1)Z = 0$, and $[X, Z] = \alpha_2(X_2)(Y + \theta Y) \neq 0$. Thus $Z \in \mathfrak{p} \cap (Z_{\mathfrak{g}}(X_1) \setminus Z_{\mathfrak{g}}(X))$. Hence we have shown $\mathrm{d}(Z_G(X)) < \mathrm{d}(Z_G(X_1))$. ∎

EXAMPLE 5.3.4. The following homogeneous space of reductive type is an elliptic orbit which admits a uniform lattice and which does not carry any invariant Riemannian metric: $SU(2n,2)/U(2n,1)$, $SO(2n,2)/U(n,1)$, and $SO(4,3)/SO(4,1) \times SO(2)$.

5.4. semisimple symmetric space

Let us recall the notion of ϵ-family of semisimple symmetric spaces introduced by Oshima-Sekiguchi. We also review some necessary notions of semisimple symmetric pair for the benefit of the reader. Let $\mathfrak{g}$ be a semisimple Lie algebra, σ be an involution of $\mathfrak{g}$, θ be a Cartan involution of $\mathfrak{g}$ commuting with σ. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{h} + \mathfrak{q}$ be direct sum decomposition corresponding θ, σ respectively. Put $\mathfrak{h}^a := \{X \in \mathfrak{g} : \sigma\theta(X) = X\} = \mathfrak{h} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p}$. Then $(\mathfrak{g}, \mathfrak{h}^a)$ is called the associated symmetric pair of $(\mathfrak{g}, \mathfrak{h})$. Note that $(\mathfrak{h}^a)^a = \mathfrak{h}$. Take a maximal abelian subspace $\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}$ of $\mathfrak{p} \cap \mathfrak{q}$. Then $\Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}) := \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p},\mathfrak{q}})$ satisfies the axiom of root system and is called the restricted root system of $(\mathfrak{g}, \mathfrak{h})$. The signature of a restricted root is a map (m^+, m^-): $\Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}) \to \mathbf{N} \times \mathbf{N}$ defined by $m^+(\lambda) := \dim \mathfrak{h}^a \cap \mathfrak{g}(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}; \lambda)$, $m^-(\lambda) := \dim \mathfrak{g}(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}; \lambda) - m^+(\lambda)$. A map ϵ: $\Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}) \cup \{0\} \to \{1, -1\}$ is called a signature of $\Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}})$ if ϵ is a semigroup homomorphism with $\epsilon(0) = 1$ (see **[O-S2]** (1.9.3.1)). Note that any map $\Psi \to \{1, -1\}$ is uniquely extended to a signature, where Ψ is a fundamental system for $\Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}})$. To a signature ϵ of $\Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}})$, we associate an involution σ_ϵ by $\sigma_\epsilon(X) := \epsilon(\lambda)\sigma(X)$ if $X \in \mathfrak{g}(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}; \lambda)$,

$\lambda \in \Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}) \cup \{0\}$. Then σ_ϵ defines a symmetric pair $(\mathfrak{g}, \mathfrak{h}_\epsilon)$. The set $F((\mathfrak{g}, \mathfrak{h})) := \{(\mathfrak{g}, \mathfrak{h}_\epsilon) : \epsilon$ is a signature of $\Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}})\}$ is called a ϵ-family of symmetric pairs ([**O-S2**] §6). Among ϵ-family, there is a distinguished symmetric pair called *basic* characterized by,

$$m^+(\lambda) \geq m^-(\lambda) \text{ for any } \lambda \in \Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}) \text{ such that } \frac{\lambda}{2} \notin \Sigma(\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}).$$

It is known that there exists a basic symmetric pair of $F = F((\mathfrak{g}, \mathfrak{h}))$ unique up to isomorphisms ([**O-S2**] Proposition 6.5). If the basic symmetric pair of F is a Riemannian symmetric pair, then $m^- \equiv 0$ and F is K_ϵ-family in the sense of [**O-S1**]. Typical examples of basic symmetric pairs are $(\mathfrak{g}, \mathfrak{k})$ (Riemannian symmetric pair), $(\mathfrak{g}, \mathfrak{g})$ (trivial case), $(\mathfrak{g} + \mathfrak{g}, \text{diag}\, \mathfrak{g})$, $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g})$, $(\mathfrak{u}(p,q;\mathbb{F}), \mathfrak{u}(m;\mathbb{F}) + \mathfrak{u}(p-m,q;\mathbb{F}))$ $(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$, whose associated symmetric pair are $(\mathfrak{g}, \mathfrak{g})$, $(\mathfrak{g}, \mathfrak{k})$, $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$, $(\mathfrak{u}(p,q;\mathbb{F}), \mathfrak{u}(m,q;\mathbb{F}) + \mathfrak{u}(p-m;\mathbb{F}))$, respectively. Now we are ready to state our application of Theorem (5.1.2) to semisimple symmetric spaces:

COROLLARY 5.4.1. *If an irreducible symmetric space G/H admits a uniform lattice, then the associated symmetric pair $(\mathfrak{g}, \mathfrak{h}^a)$ is basic in ϵ-family $F((\mathfrak{g}, \mathfrak{h}^a))$.*

The proof of Corollary (5.4.1) is derived from Theorem (5.1.2) combined with (1),(2) of the following lemma.

LEMMA 5.4.2. *With notations as above, let $(\mathfrak{g}, \mathfrak{h})$ be basic in the ϵ-family $F = F((\mathfrak{g}, \mathfrak{h}))$ and $(\mathfrak{g}, \mathfrak{h}_\epsilon)$ be not basic in F. Then we have*

1) $\mathfrak{a}(H^a) \sim \mathfrak{a}({H_\epsilon}^a)$ *by an element of* $W(\mathfrak{g}, \mathfrak{a})$.

2) $\mathrm{d}(H^a) > \mathrm{d}({H_\epsilon}^a)$.

3) $\mathbb{R}\text{-rank}(H) = \mathbb{R}\text{-rank}(G/H^a) \leq \mathbb{R}\text{-rank}(G/{H_\epsilon}^a) = \mathbb{R}\text{-rank}(H_\epsilon)$.

PROOF: (1) is clear because $\mathfrak{a}_{\mathfrak{p},\mathfrak{q}}$ is a maximally split abelian subspace of ${H_\epsilon}^a$ as well as of H^a. The proof of (2) and (3) is based on the classification in [**O-S2**](see also [**Be**]). They are trivial if H is compact, because $H^a = G$ in this case. If H is noncompact, we can check them by using Table V; Table I and (1.14-16) in [**O-S2**]. ($\mathfrak{h}$ of $D^1_{l,A}$ in Table I there should read $\mathfrak{so}(l, \mathbb{C})$.) ∎

Here is a list of $G/{H_\epsilon}^a$ which does not admit a uniform lattice from Corollary (5.4.1). We omitted here the cases where $\mathbb{R}\text{-rank}\, G = \mathbb{R}\text{-rank}\, {H_\epsilon}^a$ (see Fact (5.1.1)(1)). In particular, H^a is necessarily noncompact. We also omitted the cases treated in §5.2, namely, an indefinite Grassmann manifold $G/H = U(i+j, k+l; \mathsf{F})/U(i, k; \mathsf{F}) \times U(j, l; \mathsf{F})$, ($\mathsf{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$). (We can find the same neccesary condition with that of Example (5.2.1) if we apply Corollary (5.4.1).)

$\mathfrak{g}$	${\mathfrak{h}_\epsilon}^a$	$\mathfrak{h}^a$
$\mathfrak{sl}(2l, \mathbb{R})$	$\mathfrak{so}(l, l)$	$\mathfrak{sp}(l, \mathbb{R})$
$\mathfrak{su}^*(4l)$	$\mathfrak{so}^*(4l)$	$\mathfrak{sp}(l, l)$
$\mathfrak{su}(2l, 2l)$	$\mathfrak{so}^*(4l)$	$\mathfrak{sp}(l, l)$
$\mathfrak{sp}(2l, \mathbb{R})$	$\mathfrak{u}(l, l)$	$\mathfrak{sp}(l, \mathbb{C})$
$\mathfrak{so}(2l, 2l)$	$\mathfrak{so}(2l, \mathbb{C})$	$\mathfrak{u}(l, l)$
$\mathfrak{so}^*(4l+4)$	$\mathfrak{so}^*(4p+2) + \mathfrak{so}^*(4l-4p+2)$	$\mathfrak{so}^*(2) + \mathfrak{so}^*(4l+2)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(4, \mathbb{R})$	$\mathfrak{f}_{4(4)}$
$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(4, 2) + \mathfrak{su}(2)$	$\mathfrak{so}^*(10) + \sqrt{-1}\mathbb{R}$
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(4, 4)$	$\mathfrak{e}_{6(2)} + \sqrt{-1}\mathbb{R}$
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}^*(8)$	$\mathfrak{so}^*(12) + \mathfrak{su}(2)$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{su}(6, 2)$	$\mathfrak{e}_{6(-14)} + \sqrt{-1}\mathbb{R}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{su}(6, 2)$	$\mathfrak{e}_{6(-14)} + \sqrt{-1}\mathbb{R}$
$\mathfrak{e}_{8(8)}$	$\mathfrak{so}^*(16)$	$\mathfrak{e}_{7(-5)} + \mathfrak{su}(2)$
$\mathfrak{so}(2l+2, \mathbb{C})$	$\mathfrak{so}(2p+1, \mathbb{C}) + \mathfrak{so}(2l-2p+1, \mathbb{C})$	$\mathfrak{so}(2l+1, \mathbb{C})$
$\mathfrak{e}_{6,\mathbb{C}}$	$\mathfrak{sp}(4, \mathbb{C})$	$\mathfrak{f}_{4,\mathbb{C}}$

Table 5.4.3.

There is still a room for applications of Theorem (5.1.2). Here is a list of some other typical examples of G/H which does not admit a uniform lattice. For most of parameters below, G' stands for a reductive group satisfying the conditions in Theorem (5.1.2). We have no intention to make a complete list in Table 5.4.4.

$\mathfrak{g}$	$\mathfrak{h}$		$\mathfrak{g}'$
$\mathfrak{g}(n)$	$\mathfrak{g}(p)+\mathfrak{g}(q)$	$p+q \leq n$, $pq>0$	$\mathfrak{g}(p+q)$
$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{sp}(m,\mathbb{R})$	$0<2m\leq n-2$	$\mathfrak{so}(m,n-m)$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sp}(m,\mathbb{C})$	$0<2m\leq n-1$	$\mathfrak{u}(m,n-m)$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{so}(m,\mathbb{C})$	$0<2m\leq n$	$\mathfrak{u}(\frac{m}{2},n-\frac{m}{2})$
$\mathfrak{sl}(2l,\mathbb{C})$	$\mathfrak{u}(l,l)$		$\mathfrak{sp}(l,\mathbb{C})$
$\mathfrak{so}^*(2l,\mathbb{C})$	$\mathfrak{u}(l,n-l)$	$3l\leq 2n\leq 6l$, $n\geq 3$	$\mathfrak{so}^*(4l+2)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(l,\mathbb{C})$	$0<2l\leq n$	$\mathfrak{u}(l,n-l)$
$\mathfrak{su}^*(2n)$	$\mathfrak{so}^*(2l)$	$1<l\leq n$	$\mathfrak{sp}([\frac{l}{2}],n-[\frac{l}{2}])$

Table 5.4.4.

We give here some remarks on Table (5.4.4).

1) In the first line, $\mathfrak{g}(n)$ stands for one of the following classical Lie algebras: $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{so}^*(2n)$, $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{sp}(n,\mathbb{C})$. We also remark that $\mathfrak{g}'=\mathfrak{g}(p+q)$ should be modified by $\mathfrak{g}(p+q-1)$ if $\mathfrak{g}(n)=\mathfrak{so}^*(2n)$ or $\mathfrak{so}(n,\mathbb{C})$ and if both p and q are odd integers.

2) As for $(\mathfrak{g},\mathfrak{h})=(\mathfrak{so}^*(2l,\mathbb{C}),\mathfrak{u}(l,n-l))$, the choice $\mathfrak{g}'=\mathfrak{so}^*(4l+2)$ is valid for $2l<n$. If $3l\leq 2n<4l$ or $n=2l$, then we can choose $\mathfrak{g}'=\mathfrak{so}^*(4n-4l+2)$, $\mathfrak{g}'=\mathfrak{so}^*(2n)$, respectively.

The condition $3l\leq 2n\leq 6l$ looks strange. It is interesting to note that if $(n,l)=(4,1)$ or $(4,3)$, $G/H=SO^*(2n)/U(l,n-l)$ admits a uniform lattice.

3) As for $(\mathfrak{g},\mathfrak{h})=(\mathfrak{sp}(n,\mathbb{R}),\mathfrak{sp}(l,\mathbb{C}))$, we have to use Example (4.11) in **[Ko]** if $n=2l$ instead of Theorem (5.1.2).

REMARK 5.4.5. It is likely that a complex irreducible semisimple symmetric space

$G_{\mathbb{C}}/H_{\mathbb{C}}$ admits a uniform lattice if and only if $G_{\mathbb{C}}/H_{\mathbb{C}}$ is locally isomorphic to a group manifold. From Fact (3.3) and Tables (5.4.3) and (5.4.4), we are left with $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = (\mathfrak{gl}(2n, \mathbb{C}), \mathfrak{sp}(n, \mathbb{C}))$, $(\mathfrak{so}(2n+1, \mathbb{C}), \mathfrak{so}(2n, \mathbb{C}))$, $(\mathfrak{e}_{6,\mathbb{C}}, \mathfrak{f}_{4,\mathbb{C}})$.

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