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Campanato type estimates for solutions of difference-elliptic partial differential equations with constant coefficients

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Abstract. Difference-elliptic partial differential equations are discussed and Campanato type estimates are obtained for solutions of the equations

1. Introduction

In treating the regularity of solutions of nonlinear elliptic and parabolic partial differntial equations, it has been known that Campamato type estimates for solutions of the corresponding linear equations play a fundamental role. Such estimates have been established by Campanato ([1] and [2]) and Da Prato ([3]) and have a lot of applications in the theory of elliptic and parabolic partial differential equations and of the calculus of variations (refer to [4] and [6]).

The aim of this paper is to obtain Campanato type estimates for solutions of difference-elliptic partial differential equations with constant coefficients. In contructing Morse flows for a functional in the calculus of variations, we think a time-discrete apporoximation of the evolution equations will play an essential role (refer to [7]) and such estimates represented as in this paper will be fundamental.

Let Ω be a bounded open set in the Euclidean space R^m, $m \geq 2$, $u = (u^1, u^2, \ldots, u^M)$ be a mapping: $\Omega \to R^M, M \geq 1$ and $Du = (D_1 u, D_2 u, \ldots, D_m u)$, $D_\alpha u = \partial u / \partial x^\alpha$ $(1 \leq \alpha \leq m)$ be the gradient of u. Let T be a positive number arbitrarily given and set $Q = (0,T) \times \Omega$. We use the usual Banach space $L_p(\Omega)$, Sobolev spaces $W_p^k(\Omega) = W_p^k(\Omega, R^M)$ and $\overset{\circ}{W}{}_p^k(\Omega) = \overset{\circ}{W}{}_p^k(\Omega, R^M)$. For vectors $u, v \in R^M$, we put $uv = \sum_{j=1}^M u^j u^j$ and $|u| = \sqrt{uu}$.

For a positive integer N, $N \geq 2$, we put $h = T/N$ and $t_n = nh$ $(0 \leq n \leq N)$. Let u_0 be a function belonging to $W_2^1(\Omega)$. We shall be concerned with a family of linear elliptic partial differential equations:

$$\frac{u_n^i - u_{n-1}^i}{h} = D_\alpha(A_{ij}^{\alpha\beta} D_\beta u_n^j) \qquad (1 \leq n \leq N) \tag{1.1}$$

for each i, $1 \leq i \leq M$. In the summation convention over repeated indices, the Greek indices run from 1 to m and the Latin ones from 1 to M. The assumption of the coefficients $A_{ij}^{\alpha\beta}$ is the following: $\{A_{ij}^{\alpha\beta}\}$ $(1 \leq \alpha, \beta \leq m, 1 \leq i,j \leq M)$ is a constant matrix satisfying so-called Legendre-Hadamard condition with a positive constant λ :

$$A_{ij}^{\alpha\beta} \xi_\alpha \xi_\beta \eta^i \eta^j \geq \lambda |\xi|^2 |\eta|^2 \qquad \text{for } \xi = (\xi_\alpha) \in R^m \text{ and } \eta = (\eta^i) \in R^M. \tag{1.2}$$

Let f be a function belonging to $W_2^1(\Omega)$. We mean a family of weak solutions of (1.1) with an initial datum u_0 by a family $\{u_n\}$ $(1 \leq n \leq N)$ of functions $u_n \in W_2^1(\Omega)$ which satisfy

$$\int_\Omega \frac{u_n - u_{n-1}}{h} \varphi dx + \int_\Omega A_{ij}^{\alpha\beta} D_\beta u_n^j D_\alpha \varphi^i dx = 0 \quad \text{for any } \varphi = (\varphi^i) \in \overset{\circ}{W}{}_2^1(\Omega). \tag{1.3}$$

Morever, if the condition

$$u_n - f \in \overset{\circ}{W}{}_2^1(\Omega) \quad (1 \le n \le N) \tag{1.4}$$

is satisfied, we call $\{u_n\}$ a family of weak solutions with an initial datum u_0 and a boundary datum f.

For a family $\{u_n\}(1 \le n \le N)$ satisfying $u_n \in W_2^1(\Omega)$, we define a mapping $u_h(t,\cdot)$: $t \in [0,T] \to u_h(t,\cdot) \in W_2^1(\Omega)$ as follows:

$$\begin{aligned} &u_h(0,\cdot) = u_0(\cdot), \\ &u_h(t,\cdot) = u_n(\cdot) \qquad \text{for } t_{n-1} < t \le t_n \qquad (1 \le n \le N). \end{aligned} \tag{1.5}$$

If $\{u_n\}(1 \le n \le N)$ is a family of weak solutions of (1.1) satisfying (1.4), we then call u_h, defined by (1.5), a weak solution of (1.1) with an initial datum u_0 and a boundary datum f and for simplicity we call u_h a weak solution of (1.1). We here recall some standard notations: For a point $z_0 = (t_0, x_0) \in Q$, we put

$$\begin{aligned} &B_R(x_0) = \{x \in R^m : |x - x_0| < R\}, \\ &Q_{r,s}(z_0) = \{z = (t,x) \in Q : |x - x_0| < r, t_0 - s < t < t_0\}, \\ &Q_\rho(z_0) = Q_{\rho,\rho^2}(z_0). \end{aligned} \tag{1.6}$$

In the above notation of $B_R(x_0)$, $Q_{r,s}(z_0)$ and $Q_\rho(z_0)$, the centre x_0 and z_0 will be abbreviated when no confusion may arise. For $z_i = (t_i, x_i)$ $(i = 1,2)$, we introduce the parabolic metric

$$\delta(z_1, z_2) = max\{|t_1 - t_2|^{1/2}, |x_1 - x_2|\} \tag{1.7}$$

and for a measurable set A in R^k we denote the k-dimensional measure of A by $|A|$. For a positive r and u_h, we shall use the notation

$$\bar{u}_{h,r}(t_{n_0}, x_0) = \frac{1}{|Q_r|} \int_{Q_r(t_{n_0}, x_0)} u_h(z)dz. \tag{1.8}$$

We remark that for a positive number l we denote by $[l]$ the greatest non-positive integer not greater than l. The same letter C will be used to denote different constants depending on the same set of arguments.

Now let h_0 be an arbitrarily given and fixed positive number sufficiently small. From now on, we take N sufficiently large and assume that $h(= T/N)$ in the system (1.1) is smaller than h_0, i.e., $0 < h < h_0$. Let L be a positive number with $L > 2$ and k be a positive integer with $2k > m$. We put

$$\begin{aligned} &\widetilde{\Omega_{h_0}} = \{x \in \Omega \ : \ \text{dist}(x, \partial\Omega) \ge \sqrt{(C(m)+1)h_0}\}, \\ &\widetilde{Q_{h_0}} = [(C(m)+1)h_0, T] \times \widetilde{\Omega_{h_0}}, \end{aligned} \tag{1.9}$$

where $C(m)$ is a positive number defined by

$$C(m) = \max\{8(k+2)/3, 4L/(L-2)\} \tag{1.10}$$

and $\text{dist}(x, \partial\Omega)$ is the Eucledian distance between x and $\partial\Omega$.

Then our main result is the following, the proof of which will be given in Chapter 3.

Theorem. *Let u_h be a weak solution of (1.1). Then there exist positive constants C and α, $0 < \alpha < 1$, independent of h and u_h such that the estimate*

$$\int_{Q_r(t_n,x)} |Du_h|^2 dz \le C\left(\frac{r}{\rho}\right)^{m+2} \int_{Q_\rho(t_n,x)} |Du_h|^2 dz + C\rho^{m+2\alpha} \tag{1.11}$$

holds for all $(t_n, x) \in \widetilde{Q_{h_0}}(1 \le n \le N), r$ and ρ satisfying $0 < r < \rho < \sqrt{h_0}$.

In the paper [7] the Hölder estimates of solutions for a difference-elliptic partial differential equation are obtained and the same technique used in this paper has been represented.

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2. Some Lemmata

Let $u_h = u_h(z)$ be such a step function defined as in (1.5). In Lemma 2.1 and 2.2 we don't assume u_h to be a weak solution of (1.2).

For the gradient operator D and a positive integer k, $k \ge 2$, we define an operator D^k by $D^k = DD^{k-1}$, where $D^1 = D$.

Lemma 2.1 (Sobolev inequality). *Let $u_h(t,\cdot)$ belong to $W_p^k(\Omega), p \ge 1$, for each t, $0 \le t \le T$. If $kp > m$ is satisfied, for each positive constant ρ satisfying $\rho^2 > hL/(L-2)$ with $L > 2$ there exists a positive constant $C(\rho)$ depending on ρ such that the inequality*

$$\sup_{Q_\rho(t_{n_0},x_0)} |u_h| \le C(\rho)\left(\int_{Q_\rho(t_{n_0},x_0)} |D^k u_h|^p dz + \int_{Q_\rho(t_{n_0},x_0)} |D^k \bar{\partial}_t u_h|^p dz\right)^{1/p} \tag{2.1}$$

holds for any $(t_{n_0}, x_0) \in Q$, $1 \le n_0 \le N$, where $\bar{\partial}_t u_h(t)$ is the mapping defined by

$$\bar{\partial}_t u_h(t) = (u(t) - u(t-h))/h$$

for $h = t_1 \le t \le T$.

The next lemma is connected with estimating an oscillation of u_h. This is known to hold for functions with continuous time variables ([2] and [3]). We recall that $\bar{u}_{h,r}(t_n, x)$ is the function defined in (1.8).

Lemma 2.2. *Let $u_h(t,\cdot)$ belong to $L_p(\Omega)$, $p \ge 1$, for each t, $0 \le t \le T$. If the function u_h satisfies*

$$\int_{Q_r(t_n,x)} |u_h - \bar{u}_{h,r}(t_n,x)|^p dz \le Cr^{m+2+p\alpha} \tag{2.2}$$

for all $Q_r(t_n,x), 1 \le n \le N$, with uniform positive constants C and $\alpha, 0 < \alpha < 1$, then there exists a positive constants $\widetilde{C}$ independent of h and u_h such that the estimate

$$|u_h(t_n,x) - u_h(t_{n'},x')| \le \widetilde{C}[\delta((t_n,x),(t_{n'},x'))]^\alpha \tag{2.3}$$

holds for each (t_n, x) and $(t_{n'}, x') \in Q$ satisfying $\delta((t_n,x),(t_{n'},x')) \le \frac{1}{2}\min(dist(x,\partial\Omega), dist(x',\partial\Omega), \sqrt{t_n}, \sqrt{t_{n'}})$.

Next we shall state two fundamental properties for a weak solution of (1.1).

The inequality (2.4) in Lemma 2.3 is so called Poincarè inequality for step functions with respect to time variables. M.Struwe([10]) has shown such an inequality for weak solutions of parabolic differential equations with the quadratic nonlinearity of gradients .

Lemma 2.3 (Poincarè inequality). *Let* u_h *be a weak solution of* (1.1). *Then there exists a positive constant* C *independent of* h *and* u_h *such that*

$$\int_{Q_r(t_{n_0},x_0)} |u_h(z) - \bar{u}_{h,r}(t_{n_0},x_0)|^2 dz \quad \le \quad Cr^2 \int_{Q_r(t_{n_0},x_0)} |Du_h(z)|^2 dz \tag{2.4}$$

holds for any $(t_{n_0}, x_0) \in Q$, $1 \le n_0 \le N$, *and positive number* r.

For the proof of Lemma 2.1, 2.2 and 2.3, we can refer to Appendix.

Next we shall show that so-called Caccioppoli type inequality holds for a weak solution of (1.1) .

Lemma 2.4 (Caccioppoli type estimate). *Let* u_h *be a weak solution of* (1.1). *Then there exists a positive constant* C *independent of* h *and* u_h *such that an inequality of Caccioppoli type*

$$\int_{Q_{r,s}(t_{n_0},x_0)} |Du_h|^2 dz \le C\left[(\rho - r)^{-2} + (\tau - s)^{-1}\right] \int_{Q_{\rho,\tau}(t_{n_0},x_0)} |u_h|^2 dz \tag{2.5}$$

holds for all $Q_{r,s}(t_{n_0}, x_0)$ *and* $Q_{\rho,\tau}(t_{n_0}, x_0), x_0 \in \Omega$, $1 \le n_0 \le N$, *satisfying* $[\tau/h] - [s/h] \ge 2$ *and* $0 < r < \rho$.

Proof . Let $\eta(x) \in C_0^\infty(B_\rho(x_0))$ be a cut-off function such that $0 \le \eta \le 1$, $\eta = 1$ on $B_r(x_0)$ and $|D\eta| \le 2/(\rho - r)$. Moreover, we define a function $\sigma(t)$ on $[t_{n_0} - \tau, t_{n_0}]$ as follows:

$$\sigma(t) = \sigma_n \qquad for \qquad t_{n-1} < t \le t_n \qquad (1 \le n \le N), \tag{2.6}$$

$$\sigma_n = \begin{cases} 1, & n_0 - [s/h] \le n \le n_0, \\ \{n - n_0 + [\tau/h] - 1\} / \{[\tau/h] - 1 - [s/h]\}, & n_0 - [\tau/h] + 1 \le n \le n_0 - [s/h] - 1, \\ 0, & n \le n_0 - [\tau/h]. \end{cases}$$

Using a testing function $\varphi = \sigma\eta^2 u_h$ in the identity (1.3) and integrating the resultant equality over t in $[t_{n_0} - \tau, t_{n_0}]$, we obtain

$$\begin{aligned} &\frac{1}{h}\int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho(x_0)} \sigma(t)\eta^2(x)u_h(t,x)(u_h(t,x) - u_h(t-h,x))dxdt \\ &+ \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho(x_0)} \sigma(t)A_{ij}^{\alpha\beta} D_\beta u_h^j(t,x) D_\alpha(\eta^2(x)u_h^i(t,x))dxdt = 0. \end{aligned} \tag{2.7}$$

For brevity, we put the first and second term of the left-hand of (2.7) as L_1 and L_2, respectively. By virtue of the definition of $\sigma(t)$, L_1 reduces to

$$\begin{aligned} L_1 &= \frac{1}{h}\int_{t_{n_0}-[\tau/h]}^{t_{n_0}} \int_{B_\rho(x_0)} \sigma(t)\eta^2(x)u_h(t,x)\left(u_h(t,x) - u_h(t-h,x)\right)dxdt \\ &= \sum_{n=n_0-[\tau/h]+2}^{n_0} \int_{B_\rho(x_0)} \sigma_n\eta^2(x)u_n(x)\left(u_n(x) - u_{n-1}(x)\right)dx \\ &= \sum_{n=n_0-[s/h]+1}^{n_0} \int_{B_\rho(x_0)} \eta^2(x)u_n(x)\left(u_n(x) - u_{n-1}(x)\right)dx \\ &+ \sum_{n=n_0-[\tau/h]+2}^{n_0-[s/h]} \int_{B_\rho(x_0)} \sigma_n\eta^2(x)u_n(x)\left(u_n(x) - u_{n-1}(x)\right)dx. \end{aligned}$$

We here remark that if $[s/h]=0$,

$$L_1 = \sum_{n=n_0-[\tau/h]+2}^{n_0} \int_{B_\rho(x_0)} \sigma_n \eta^2(x) u_n(x)\,(u_n(x)-u_{n-1}(x))\,dx.$$

By using an inequality

$$u_n(u_n - u_{n-1}) \geq (|u_n|^2 - |u_{n-1}|^2)/2,$$

we infer

$$\begin{aligned} L_1 \geq{}& \frac{1}{2} \sum_{n=n_0-[s/h]+1}^{n_0} \int_{B_\rho(x_0)} \eta^2(x)\left(|u_n(x)|^2 - |u_{n-1}(x)|^2\right) dx \\ &+ \frac{1}{2} \sum_{n=n_0-[\tau/h]+2}^{n_0-[s/h]} \int_{B_\rho(x_0)} \sigma_n \eta^2(x)\left(|u_n(x)|^2 - |u_{n-1}(x)|^2\right) dx \\ ={}& \frac{1}{2}\int_{B_\rho(x_0)} \eta^2(x)|u_{n_0}(x)|^2 dx - \frac{1}{2}\int_{B_\rho(x_0)} \eta^2(x)|u_{n_0-[s/h]}(x)|^2 dx \\ &+ \frac{1}{2} \sum_{n=n_0-[\tau/h]+2}^{n_0-[s/h]} \int_{B_\rho(x_0)} \sigma_n \eta^2(x)\left(|u_n(x)|^2 - |u_{n-1}(x)|^2\right) dx. \end{aligned}$$

Moreover, by using an equality

$$\sigma_n(|u_n|^2 - |u_{n-1}|^2) = \sigma_n|u_n|^2 - \sigma_{n-1}|u_{n-1}|^2 - (\sigma_n - \sigma_{n-1})|u_{n-1}|^2,$$

we obtain

$$\begin{aligned} L_1 \geq{}& \frac{1}{2}\int_{B_\rho(x_0)} \eta^2(x)|u_{n_0}(x)|^2 dx - \frac{1}{2}\int_{B_\rho(x_0)} \eta^2(x)|u_{n_0-[s/h]}(x)|^2 dx \\ &+ \frac{1}{2} \sum_{n=n_0-[\tau/h]+2}^{n_0-[s/h]} \int_{B_\rho(x_0)} \eta^2(x)\left(\sigma_n|u_n(x)|^2 - \sigma_{n-1}|u_{n-1}(x)|^2\right) dx \\ &- \frac{1}{2} \sum_{n=n_0-[\tau/h]+2}^{n_0-[s/h]} (\sigma_n - \sigma_{n-1}) \int_{B_\rho(x_0)} \eta^2(x)|u_{n-1}(x)|^2 dx \end{aligned}$$

(2.8)

$$= \frac{1}{2}\int_{B_\rho(x_0)} \eta^2(x)|u_{n_0}(x)|^2 dx - \frac{1}{2} \sum_{n=n_0-[\tau/h]+2}^{n_0-[s/h]} (\sigma_n - \sigma_{n-1}) \int_{B_\rho(x_0)} \eta^2(x)|u_{n-1}(x)|^2 dx.$$

According to the definition (2.6) of σ_n, we have for $n_0 - [\tau/h] + 2 \leq n \leq n_0 - [s/h]$ that

$$\sigma_n - \sigma_{n-1} \leq 3h/(\tau - s).$$

In fact , if $\tau - s < 3h$,

$$\sigma_n - \sigma_{n-1} \leq 1 < 3h/(\tau - s)$$

and if $\tau - s \geq 3h$,

$$\begin{aligned}\sigma_n - \sigma_{n-1} &\leq 1/(\tau/h - 2 - s/h) = h/(\tau - s - 2h) \\ &\leq h/(\tau - s - 2(\tau - s)/3) = 3h/(\tau - s).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\sum_{n=n_0-[\tau/h]+2}^{n_0-[s/h]} (\sigma_n - \sigma_{n-1}) \int_{B_\rho(x_0)} \eta^2(x)|u_{n-1}(x)|^2 dx \\ &\leq 3(\tau - s)^{-1} h \sum_{n=n_0-[\tau/h]+1}^{n_0-[s/h]-1} \int_{B_\rho(x_0)} \eta^2(x)|u_n(x)|^2 dx \\ &\leq 3(\tau - s)^{-1} \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho(x_0)} \eta^2(x)|u_h(t,x)|^2 dx dt, \qquad (2.9)\end{aligned}$$

so that from (2.8) and (2.9) we obtain

$$L_1 \geq \frac{1}{2}\int_{B_\rho(x_0)} \eta^2(x)|u_{n_0}(x)|^2 dx - \frac{3}{2}(\tau - s)^{-1} \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho(x_0)} \eta^2(x)|u_h(t,x)|^2 dx dt.$$

On the other hand, noting that

$$\begin{aligned}L_2 = &\int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho(x_0)} \sigma(t) A_{ij}^{\alpha\beta} D_\beta\big(\eta(x)u_h^j(t,x)\big) D_\alpha\big(\eta(x)u_h^i(t,x)\big) dx dt \\ &- \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho(x_0)} \sigma(t) A_{ij}^{\alpha\beta} D_\beta\eta(x) D_\alpha\eta(x) u_h^j(t,x) u_h^i(t,x) dx dt\end{aligned}$$

and that by Legendre-Hadamard condition on $\{A_{ij}^{\alpha\beta}\}$ we have

$$\begin{aligned}&\lambda \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho(x_0)} \sigma(t) |D(\eta(x)u_h(t,x))|^2 dx dt \\ &\leq \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho(x_0)} \sigma(t) A_{ij}^{\alpha\beta} D_\beta\big(\eta(x)u_h^j(t,x)\big) D_\alpha\big(\eta(x)u_h^i(t,x)\big) dx dt,\end{aligned}$$

we obtain for some positive constant C that

$$\begin{aligned}&\frac{1}{2}\int_{B_\rho(x_0)} \eta^2(x)|u_{n_0}(x)|^2 dx + \lambda \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho(x_0)} \sigma(t)|D(\eta(x)u_h(t,x))|^2 dx dt \\ \leq &\frac{3}{2}(\tau - s)^{-1} \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho(x_0)} \eta^2(x)|u_h(t,x)|^2 dx dt + C(\rho - r)^{-2} \int_{t_{n_0}-\tau}^{t_{n_0}} \int_{B_\rho(x_0)} |u_h(t,x)|^2 dx dt,\end{aligned}$$

which yields the required estimate

$$\begin{aligned}&\int_{Q_{r,s}(t_{n_0},x_0)} |Du_h|^2 dz \\ &\leq \frac{3}{2}\lambda^{-1}(\tau - s)^{-1} \int_{Q_{\rho,\tau}(t_{n_0},x_0)} |u_h|^2 dz + C\lambda^{-1}(\rho - r)^{-2} \int_{Q_{\rho,\tau}(t_{n_0},x_0)} |u_h|^2 dz.\end{aligned}$$

The inequality of the type (2.5) holds for the spatial higher derivatives in the following form.

Lemma 2.5. *Let u_h be a weak solution of* (1.1). *Then for each positive integer k there exists a positive constant C independent of h and u_h such that*

$$\int_{Q_{r,s}(t_{n_0},x_0)} |D^k u_h|^2 dz \le C\left[(\rho - r)^{-2} + (\tau - s)^{-1}\right]^k \int_{Q_{\rho,\tau}(t_{n_0},x_0)} |u_h|^2 dz \tag{2.10}$$

holds for all $Q_{r,s}(t_{n_0}, x_0)$ and $Q_{\rho,\tau}(t_{n_0}, x_0)$, $x_0 \in \Omega, 1 \le n_0 \le N$, satisfying $\tau - s \ge 2kh$ and $0 < r < \rho$.

Proof. For each integer j, $0 \le j \le k$, we put

$$\rho_j = r + (k-j)(\rho - r)/k, \quad \tau_j = s + (k-j)(\tau - s)/k, \quad Q_{\rho_j,\tau_j} = Q_{\rho_j,\tau_j}(t_{n_0}, x_0).$$

Now noting $(\tau - s)/k \ge 2h$ and using Lemma 2.4 , we have

$$\int_{Q_{\rho_1,\tau_1}} |Du_h|^2 dz \le C\left[(k/(\rho - r))^2 + k/(\tau - s)\right] \int_{Q_{\rho_0,\tau_0}} |u_h|^2 dz.$$

By using the difference quotient method with respect to the spatial variables and calculating as in the proof of Lemma 2.4, we obtain

$$\int_{Q_{\rho_2,\tau_2}} |D^2 u_h|^2 dz \le C\left[(k/(\rho - r))^2 + k/(\tau - s)\right] \int_{Q_{\rho_1,\tau_1}} |Du_h|^2 dz.$$

Similarly as above, we have for $0 \le j \le k$ that

$$\int_{Q_{\rho_j,\tau_j}} |D^j u_h|^2 dz \le C\left[(k/(\rho - r))^2 + k/(\tau - s)\right] \int_{Q_{\rho_{j-1},\tau_{j-1}}} |D^{j-1} u_h|^2 dz.$$

By repeating the above argument, we have

$$\int_{Q_{r,s}(t_{n_0},x_0)} |D^k u_h|^2 dz \le C\left[(k/(\rho - r))^2 + k/(\tau - s)\right]^k \int_{Q_{\rho,\tau}(t_{n_0},x_0)} |u_h|^2 dz,$$

which is the required inequality.

Lemma 2.6. *Let u_h be a weak solution of system* (1.1) . *Then there exist positive constants C and α, independent of h and u_h such that the estimate*

$$|u_{n'}(x') - u_n(x)| \le C[\delta((t_{n'}, x'), (t_n, x))]^\alpha \tag{2.11}$$

holds for all (t_n, x) and $(t_{n'}, x') \in Q$ satisfying $x, x' \in \{x \in \Omega : dist(x, \partial\Omega) \ge \sqrt{C(m)h_0}\}$, $t_n, t_{n'} \in [C(m)h_0, T]$ and $\delta((t_{n'}, x'), (t_n, x)) \le \frac{1}{2} min(dist(x, \partial\Omega), dist(x', \partial\Omega), \sqrt{t_n}, \sqrt{t_{n'}})$.

Proof. We take ρ such that $\rho^2 > 4hL/(L-2)$, where L is a positive number with $L > 2$. Let k be a positive integer satisfying $2k > m$. We then have by Lemma 2.1 that for all $r < \rho$

$$\int_{Q_r(t_{n_0},x_0)} |Du_h|^2 dz \le \sup_{Q_{\rho/2}(t_{n_0},x_0)} |Du_h|^2 |Q_r| \le C(\rho) \|Du_h\|^2_{\widetilde{W}_2^k(Q_{\rho/2}(t_{n_0},x_0))} |Q_r|$$

where $\|\cdot\|_{\widetilde{W}_2^k(Q_{\rho/2})}$ is the norm defined in the right hand of (2.1) in Lemma 2.1. Moreover, noting that u_h is a weak solution of (1.1) and using Lemma 2.5, we have for $\rho^2 \geq 8(k+2)h/3$

$$\|Du_h\|^2_{\widetilde{W}_2^k(Q_{\rho/2}(t_{n_0},x_0))} \leq C(\rho)\int_{Q_\rho(t_{n_0},x_0)} |Du_h|^2 dz.$$

Hence we obtain

$$\int_{Q_r(t_{n_0},x_0)} |Du_h|^2 dz \leq C(\rho)|Q_r| \int_{Q_\rho(t_{n_0},x_0)} |Du_h|^2 dz. \tag{2.12}$$

Let us now use the dilatation argument. We shall notice two facts. At first we have that the scaled function

$$\widetilde{u}_h(s,y) := u_h(t_{n_0} + \rho^2 s, x_0 + \rho y)$$

satisfies, for each nonpositive integer $l, -[\rho^2/h] \leq l \leq 0$

$$\widetilde{u}_h(s,y) = u_{n_0+l}(x_0+\rho y) \quad \text{for} \quad (l-1)h/\rho^2 < s \leq lh/\rho^2.$$

This follows from that, for integer l

$$(l-1)h/\rho^2 < s \leq lh/\rho^2$$

is equivalent to

$$t_{n_0+l-1} < t_{n_0} + \rho^2 s \leq t_{n_0+l}.$$

Secondly, setting

$$\widetilde{u}_l(y) = u_{n_0+l}(x_0+\rho y),$$

the following is valid for each nonpositive integer $l, -[\rho^2/h] \leq l \leq 0$:

$$\int_{B_1(0)} A_{ij}^{\alpha\beta} D_\beta \widetilde{u}_l^j(y) D_\alpha \varphi^i(y) dy = -\int_{B_1(0)} \frac{\widetilde{u}_l(y) - \widetilde{u}_{l-1}(y)}{h/\rho^2}\varphi(y)dy \quad \text{for} \quad \varphi \in \overset{\circ}{W}{}_2^1(B_1(0)).$$

In fact by transforming variables: $t = t_{n_0} + \rho^2 s, x = x_0 + \rho y$, we have that for $\varphi \in \overset{\circ}{W}{}_2^1(B_1(0))$

$$\int_{B_1(0)} A_{ij}^{\alpha\beta} D_\beta \widetilde{u}_l^j(y) D_\alpha \varphi^i(y) dy = \int_{B_1(0)} A_{ij}^{\alpha\beta} D_\beta u_{n_0+l}^j(x_0+\rho y) D_\alpha \varphi^i(y) dy$$
$$= \rho^{2-m} \int_{B_\rho(x_0)} A_{ij}^{\alpha\beta} D_\beta u_{n_0+l}^j(x) D_\alpha \widetilde{\varphi}^i(x) dx,$$

where $\widetilde{\varphi}(x) := \varphi(\frac{x-x_0}{\rho})$. Noting that $\widetilde{\varphi}(\cdot) \in \overset{\circ}{W}{}_2^1(B_\rho(x_0))$ and using the identity (1.3), we obtain

$$\rho^{2-m}\int_{B_\rho(x_0)} A_{ij}^{\alpha\beta} D_\beta u_{n_0+l}^j(x) D_\alpha \widetilde{\varphi}^i(x) dx = -\rho^{2-m}\int_{B_\rho(x_0)} \frac{u_{n_0+l}(x) - u_{n_0+l-1}(x)}{h}\widetilde{\varphi}(x)dx.$$

Again from changing variables: $t = t_{n_0} + \rho^2 s, x = x_0 + \rho y$, it follows

$$\rho^{2-m}\int_{B_\rho(x_0)} \frac{u_{n_0+l}(x) - u_{n_0+l-1}(x)}{h}\widetilde{\varphi}(x)dx = \rho^2 \int_{B_1(0)} \frac{u_{n_0+l}(x_0+\rho y) - u_{n_0+l-1}(x_0+\rho y)}{h}\varphi(y)dy.$$

Combining the above calculations, we have the second assertion.

Here noticing that $\rho^2 \geq 8(k+2)/3$ implies $1 \geq 8(k+2)h/3\rho^2$, we are able to estimate the L^2-norm of $D\widetilde{u_h}(s,y)$ on $Q_{r/\rho}(0,0)$ and $Q_1(0,0)$ similarly as in calculating (2.12), so that

$$\int_{Q_{r/\rho}(0,0)} |D\widetilde{u}_h(s,y)|^2 dyds \leq C(1)|Q_{r/\rho}| \int_{Q_1(0,0)} |D\widetilde{u}_h(s,y)|^2 dyds.$$

By changing variables $y = (x - x_0)/\rho, s = (t - t_{n_0})/\rho^2$, we arrive at the estimate

$$\int_{Q_r(t_{n_0},x_0)} |Du_h|^2 dz \leq C(r/\rho)^{m+2} \int_{Q_\rho(t_{n_0},x_0)} |Du_h|^2 dz. \tag{2.13}$$

holds for all $r < \rho/2$. This inequality (2.13) being valid for $\rho > r \geq \rho/2$, we conclude that (2.13) holds for all $r < \rho$.

Now we recall that $C(m)$ is a positive number defined in (1.10). For each $(t_{n_0}, x_0) \in Q$ saisfying $t_{n_0} \in [C(m)h_0, T]$ and $x_0 \in \{x \in \Omega : dist(x, \partial\Omega) \geq \sqrt{C(m)h_0}\}$ we take a positive number ρ satisfying $\rho^2 \geq C(m)h_0$ and $Q_\rho(t_{n_0}, x_0) \subset Q$. Noting that implies $\rho^{-1} \leq C(m)^{-1/2}h_0{}^{-1/2}$, we have by (2.13) that

$$\int_{Q_r(t_{n_0},x_0)} |Du_h|^2 dz \leq C(C(m)h_0)^{-(m+2)/2} r^{m+2} \int_{Q_\rho(t_{n_0},x_0)} |Du_h|^2 dz \tag{2.14}$$

holds for $(t_{n_0}, x_0) \in Q$ saisfying $t_{n_0} \in [C(m)h_0, T], x_0 \in \{x \in \Omega : dist(x, \partial\Omega) \geq \sqrt{C(m)h_0}\}$ and $r < \rho$.

On the other hand, we have the boundedness of the quantity $\int_Q |Du_h|^2\, dz$ with respect to h. In fact, substituting $\varphi = u_n - f$ into the identity (1.3) and summing the resultant inequality over n from 1 to N, we have the calculations

$$\begin{aligned}
&h \sum_{n=1}^{N} \int_\Omega A_{ij}^{\alpha\beta} D_\beta u_n^j D_\alpha u_n^i dx \\
&= h \sum_{n=1}^{N} \int_\Omega A_{ij}^{\alpha\beta} D_\beta u_n^j D_\alpha f^i dx - \sum_{n=1}^{N} \int_\Omega u_n(u_n - u_{n-1}) dx + \sum_{n=1}^{N} \int_\Omega f(u_n - u_{n-1}) dx \\
&\leq h \sum_{n=1}^{N} \int_\Omega A_{ij}^{\alpha\beta} D_\beta u_n^j D_\alpha f^i dx - \frac{1}{2}\int_\Omega |u_N|^2 dx + \frac{1}{2}\int_\Omega |u_0|^2 dx + \int_\Omega f u_N dx - \int_\Omega f u_0 dx \\
&\leq h \sum_{n=1}^{N} \int_\Omega A_{ij}^{\alpha\beta} D_\beta u_n^j D_\alpha f^i dx + \int_\Omega |f|^2 dx + \int_\Omega |u_0|^2 dx,
\end{aligned}$$

which imply the estimate

$$\int_Q |Du_h|^2 dz \leq C \int_\Omega (|u_0|^2 + |f|^2) dx + CT \int_\Omega |Df|^2. \tag{2.15}$$

Hence, using the estimate (2.14), (2.15) and Lemma 2.3, we obtain

$$\int_{Q_r(t_{n_0},x_0)} |u_h - \bar{u}_{h,r}(t_{n_0}, x_0)|^2\, dz \leq Cr^2 \int_{Q_r(t_{n_0},x_0)} |Du_h|^2\, dz \leq Cr^{m+4}.$$

Consequently, the assertion of Lemma 2.6 follows from Lemma 2.2.

We conclude this section by proving the following estimate.

Lemma 2.7. *Let u_h be a weak solution of systems (1.1). Then there exists a positive constant C independent of h and u_h such that an inequality*

(2.16)
$$\int_{B_r(x_0)} |Du_n|^2\,dx \le C\{(r/\rho)^m \int_{B_\rho(x_0)} |Du_n|^2\,dx + (\int_{B_\rho(x_0)} |\frac{u_n - u_{n-1}}{h}|^{2m/(m+2)}\,dx)^{(m+2)/m}\}$$

$(1 \le n \le N)$ holds for any $B_r(x_0)$ and $B_\rho(x_0), x_0 \in \Omega$, satisfying $B_\rho(x_0) \subset \Omega$ and $0 < r < \rho$.

Proof. We shall carry out the calculation for $m \ge 3$ and leave the analogous result in the case $m = 2$ for the reader to verify. 2^* and $(2^*)'$ shall denote the Sobolev exponent and the dual one of 2, respectively. *i.e.* $2^* = 2m/(m-2)$ and $(2^*)' = 2m/(m+2)$. Let n $(1 \le n \le N)$ and $\rho > 0$, $B_\rho(x_0) \subset \Omega$ be fixed and let $v_n \in W_2^1(B_\rho)$ be a function satisfying the relation

$$\int_{B_\rho} A_{ij}^{\alpha\beta} D_\beta v_n{}^j D_\alpha \varphi^i\,dx = 0 \qquad for\ any\ \varphi = (\varphi^1, \dots, \varphi^M) \in \overset{\circ}{W}{}_2^1(B_\rho) \tag{2.17}$$

and $v_n - u_n \in \overset{\circ}{W}{}_2^1(B_\rho)$. A fundamental estimate, due to Campanato([1]), yields that

$$\int_{B_r} |Dv_n|^2\,dx \le C(r/\rho)^m \int_{B_\rho} |Du_n|^2\,dx \tag{2.18}$$

holds for all $0 < r < \rho$, where C is a positive constant independent of r, ρ, u_n and v_n. Setting now $w_n = v_n - u_n$, we have from $w_n \in \overset{\circ}{W}{}_2^1(B_\rho)$ and the estimate (2.18) that

$$\begin{aligned} \int_{B_r} |Du_n|^2\,dx &\le 2\int_{B_r} |Dv_n|^2\,dx + 2\int_{B_r} |Dw_n|^2\,dx \\ &\le 2C(r/\rho)^m \int_{B_\rho} |Du_n|^2\,dx + 2\int_{B_r} |Dw_n|^2\,dx. \end{aligned} \tag{2.19}$$

Now we shall estimate the quantity $\int_{B_r} |Dw_n|^2\,dx$. For this purpose we subtract (1.3) from (2.17) to have

$$\int_{B_\rho} A_{ij}^{\alpha\beta} D_\alpha w_n{}^j D_\beta \varphi^i\,dx - \int_{B_\rho} \frac{u_n - u_{n-1}}{h}\varphi\,dx = 0 \tag{2.20}$$

for any $\varphi \in \overset{\circ}{W}{}_2^1(B_\rho)$. In particular, we may take $\varphi = w_n$ in (2.20), whence

$$\int_{B_\rho} A_{ij}^{\alpha\beta} D_\alpha w_n{}^j D_\beta w_n{}^i\,dx = \int_{B_\rho} \frac{u_n - u_{n-1}}{h} w_n\,dx. \tag{2.21}$$

Hölder and Sobolev inequalities yield the estimate

$$\begin{aligned} |\int_{B_\rho} \frac{u_n - u_{n-1}}{h} w_n\,dx| &\le (\int_{B_\rho} |w_n|^{2^*})^{1/2^*} (\int_{B_\rho} |\frac{u_n - u_{n-1}}{h}|^{(2^*)'}\,dx)^{1/(2^*)'} \\ &\le C(\int_{B_\rho} |Dw_n|^2\,dx)^{1/2} (\int_{B_\rho} |\frac{u_n - u_{n-1}}{h}|^{(2^*)'}\,dx)^{1/(2^*)'} \end{aligned}$$

with an absolute positive constant C. Moreover, by virtue of Young inequality we infer

$$|\int_{B_\rho} \frac{u_n - u_{n-1}}{h} w_n\,dx| \le \frac{\lambda}{2}\int_{B_\rho} |Dw_n|^2\,dx + \frac{C^2}{2\lambda}(\int_{B_\rho} |\frac{u_n - u_{n-1}}{h}|^{(2^*)'}\,dx)^{2/(2^*)'}. \tag{2.22}$$

Hence,by Legendre-Hadamard condition on $\{A_{ij}^{\alpha\beta}\}$, we conclude from (2.21) and (2.22) that

$$\int_{B_\rho} |Dw_n|^2\,dx \le \frac{C^2}{\lambda^2}(\int_{B_\rho} |\frac{u_n - u_{n-1}}{h}|^{(2^*)'}\,dx)^{2/(2^*)'} \tag{2.23}$$

holds. Thus substituting (2.23) into (2.19), we have the assertion of Lemma 2.7.

3. Proof of Theorem

For the following we fix two positive numbers ρ and h, $h < h_0$. We distinguish three cases in the relation between ρ and h:

Case 1.

$$C(m)h < \rho^2,$$

Case 2.

$$h \leq \rho^2 < C(m)h,$$

Case 3.

$$\rho^2 < h,$$

where $C(m)$ is a positive integer determined in (1.10).

Case 1. For ρ satisfying $\rho^2 > C(m)h$ we have obtained the estimate (2.13) in the proof of Lemma 2.6, from which we have the assertion.

From now on we fix $(t_{n_0}, x_0) \in \widetilde{Q_{h_0}}$ and $\rho^2 \leq h_0$.

Case 2. By virtue of (2.11) in Lemma 2.6, there exist positive numbers C and α, $0 < \alpha < 1$, independent of h and u_h such that for each $x \in \widetilde{\Omega_{h_0}}$ we have

$$|u_n(x) - u_{n-1}(x)| \leq Ch^{\alpha/2} \quad ([C(m)h_0/h] + 1 \leq n \leq N). \tag{3.1}$$

By using the inequality (3.1) and (2.16) in Lemma 2.7, we have for $x_0 \subset \widetilde{\Omega_{h_0}}$ and $n, [C(m)h_0/h]+1 \leq n \leq N$, that

$$\int_{B_r(x_0)} |Du_n(x)|^2 dx \leq C(r/\rho)^m \int_{B_\rho(x_0)} |Du_n(x)|^2\, dx + Ch^{\alpha-2}\rho^{m+2}. \tag{3.2}$$

At first, we shall show the inequality (1.11) with the restriction $0 < r < \rho/\sqrt{2}$. We here notice that $r^2 < \rho^2/2$ and $h \leq \rho^2$ imply $[\rho^2/h]h > r^2$. In fact,

$$r^2 < \rho^2/2 < ([\rho^2/h]+1)h/2 < ([\rho^2/h]+[\rho^2/h])h/2 = [\rho^2/h]h.$$

Hence we have for $t \in (t_{n_0} - r^2, t_{n_0}$ that

$$h \int_{B_\rho(x_0)} |Du_h(t,x)|^2 dx \leq \int_{Q_\rho(t_{n_0}, x_0)} |Du_h|^2\, dz. \tag{3.3}$$

Multiplying (3.2) by h and using (3.3), we obtain for $t \in (t_{n_0} - r^2, t_{n_0}$ and $r < \rho/\sqrt{2}$

$$h \int_{B_r(x_0)} |Du_h(t,x)|^2 dx \leq C(r/\rho)^m \int_{Q_\rho(t_{n_0}, x_0)} |Du_h(z)|^2 dz + C\rho^{m+2}h^{\alpha-1}. \tag{3.4}$$

Integrating both sides of (3.4) with respect to t in $(t_{n_0} - r^2, t_{n_0}$, we obtain for $r < \rho/\sqrt{2}$ that

$$h \int_{Q_r(t_{n_0}, x_0)} |Du_h(z)|^2 dz \leq C(r/\rho)^m r^2 \int_{Q_\rho(t_{n_0}, x_0)} |Du_h(z)|^2 dz + Cr^2\rho^{m+2}h^{\alpha-1}. \tag{3.5}$$

Since the assumptions $C(m)h > \rho^2$ and $0 < \alpha < 1$ imply the estimate

$$h^{-1}\rho^2 < C(m), \quad h^{\alpha-1} < (\rho^2/C(m))^{\alpha-1},$$

we conclude from (3.5) that

$$\int_{Q_r(t_{n_0},x_0)} |Du_h|^2 dz \le C(r/\rho)^{m+2} \int_{Q_\rho(t_{n_0},x_0)} |Du_h|^2 dz + C\rho^{m+2\alpha}.$$

The inequality being valid for $r \ge \rho/\sqrt{2}$, the assertion of Theorem follows in Case 2.

Case 3. We here note that the assumptions $\rho^2 < h$ and $r < \rho$ imply $r^2 < h$. Multiplying (3.2) by r^2, we obtain

$$\int_{Q_r(t_{n_0},x_0)} |Du_h(z)|^2 dz \le C(r/\rho)^{m+2} \int_{Q_\rho(t_{n_0},x_0)} |Du_h(z)|^2 dz + Cr^2 h^{\alpha-2}\rho^{m+2}.$$

Since $\rho^2 < h$ and $0 < \alpha < 1$ imply $\rho^{2(\alpha-2)} > h^{\alpha-2}$, we obtain for all $r < \rho$ that

$$\int_{Q_r(t_{n_0},x_0)} |Du_h|^2 dz \le C(r/\rho)^{m+2} \int_{Q_\rho(t_{n_0},x_0)} |Du_h|^2 dz + C\rho^{m+2\alpha}.$$

Therefore, the proof of Theorem is completed.

4. Appendix

In this chapter we shall give the proof of Lemmat 2.1, 2.2 and 2.3 stated in Chapter 2. For simplification we shall use notation: $u(t,x) = u_h(t,x)$.

For the proof of Lemma 2.1, we prepare the following Proposition.

Proposition 4.1. *Let ρ be a positive number satisfying $\rho^2 \ge hL/(L-2)$ with a positive number $L > 2$. Then for each integer j, $0 \le j \le [\rho^2/h]$, there holds at least one of two inequalities:*

$$jh > \rho^2/L, \quad [\rho^2/h]h - jh > \rho^2/L. \tag{4.1}$$

Proof. It is sufficient to prove that

$$2\rho^2/L < [\rho^2/h]h.$$

From $\rho^2 \ge hL/(L-2)$ with $L > 2$, we have

$$\begin{aligned}
&[\rho^2/h]h > (\rho^2/h - 1)h = \rho^2 - h \\
&\ge \rho^2 - (L-2)\rho^2/L = (1 - (L-2)/L)\rho^2 \\
&= 2\rho^2/L.
\end{aligned}$$

Proof of Lemma 2.1. Let $Q_\rho(t_{n_0},x_0)$ be fixed. For $(t_n,x) \in Q_\rho(t_{n_0},x_0)$ $(1 \le n \le N)$ we shall estimate the value $u(t_n,x)$. For $x \in B_\rho(x_0)$, we introduce polar coordinates (r,θ) for the spatial points y in the spherical cone $V(x)$ with the vertex x, height δ and opening α, which occurs in the cone condition of B_ρ . Let $g(s), 0 \le g(s) \le 1$, be a $C^\infty-$ function for $-\infty < s < \infty$, such that $g(s) = 1$ if $s \le \frac{1}{2}$ and $g(s) = 0$ if $s \ge 1$. Then, for $1 \le i \le N$ we have

$$u_i(x) = -\int_0^\delta \frac{\partial}{\partial r}[g(\frac{r}{\delta})u_i(r,\theta)]dr.$$

Integrating this equality with respect to $d\theta$ over the opening α, we then perform integration by parts $k-1$ times to obtain

$$u_i(x) = \frac{(-1)^k C}{(k-1)!}\int_\alpha\int_0^\delta r^{k-1}\frac{\partial^k}{\partial r^k}[g(\frac{r}{\delta})u_i(r,\theta)]drd\theta,$$

where C is a positive number. Noting $r^{k-1} = r^{k-m}r^{m-1}$ and $dy = r^{m-1}drd\theta$, we have

$$u_i(x) = \frac{(-1)^k C}{(k-1)!}\int_{V(x)} r^{k-m}\frac{\partial^k}{\partial r^k}[g(\frac{r}{\delta})u_i(r,\theta)]dy. \tag{4.2}$$

By taking j in Proposition 4.1 as $n_0 - n$, we find that for each n, $n_0 - [\rho^2/h] \le n \le n_0$, there holds at least one of two cases:

Case 1.

$$[\rho^2/h]h - (n_0 - n)h > \rho^2/L,$$

Case 2.

$$(n_0 - n)h > \rho^2/L.$$

Case 1. We remark that for each n satisfying $[\rho^2/h]h - (n_0 - n)h > \rho^2/L$, it follows

$$(t_{n-[\rho^2/hL]-1}, t_n] \times V(x) \subset Q_\rho(t_{n_0}, x_0).$$

We define a function $\sigma(t)$ on $(t_{n-[\rho^2/hL]-2}, t_n]$ as follows:

$$\begin{aligned}
\sigma(t) &= \sigma_i \qquad \text{for} \quad t_{i-1} < t \le t_i,\\
\sigma_{n-[\rho^2/hL]-1} &= 0,\\
\sigma_{n-[\rho^2/hL]} &= -[\rho^2/hL]hL/\rho^2 + 1,\\
\sigma_i &= hL/\rho^2 + \sigma_{i-1} \qquad \text{for} \quad n - [\rho^2/hL] + 1 \le i \le n.
\end{aligned}$$

Using an equality

$$\sigma_i u_i(x) - \sigma_{i-1}u_{i-1}(x) = (\sigma_i - \sigma_{i-1})u_i(x) + \sigma_{i-1}(u_i(x) - u_{i-1}(x))$$

and (4.2), we have

$$\begin{aligned}
&\sigma_i u_i(x) - \sigma_{i-1}u_{i-1}(x)\\
&= \frac{(-1)^k C}{(k-1)!}h\int_{V(x)} r^{k-m}\frac{\partial^k}{\partial r^k}[\frac{\sigma_i - \sigma_{i-1}}{h}g(\frac{r}{\delta})u_i(r,\theta)]dy\\
&+ \frac{(-1)^k C}{(k-1)!}h\int_{V(x)} r^{k-m}\frac{\partial^k}{\partial r^k}[\sigma_{i-1}g(\frac{r}{\delta})\frac{u_i(x) - u_{i-1}(x)}{h}]dy.
\end{aligned}$$

Noting we have by the definition of σ_i

$$\sigma_n u_n(x) = \sum_{i=n-[\rho^2/hL]}^{n} (\sigma_i u_i(x) - \sigma_{i-1}u_{i-1}(x)),$$

we thus obtain

$$\sigma_n u_n(x) = \frac{(-1)^k C}{(k-1)!}\int_{t_{n-[\rho^2/hL]-1}}^{t_n}\int_{V(x)} r^{k-m}\frac{\partial^k}{\partial r^k}[\frac{\sigma(t)-\sigma(t-h)}{h}g(\frac{r}{\delta})u_h(t,r,\theta)]dydt$$
$$+\frac{(-1)^k C}{(k-1)!}\int_{t_{n-[\rho^2/hL]-1}}^{t_n}\int_{V(x)} r^{k-m}\frac{\partial^k}{\partial r^k}[\sigma(t-h)g(\frac{r}{\delta})\frac{u_h(t,r,\theta)-u_h(t-h,r,\theta)}{h}]dydt.$$

Since we have $|(\sigma(t)-\sigma(t-h))/h| \le L/\rho^2$ and $|\sigma(t)| \le 1$ from the definition of $\sigma(t)$, it follows

$$|u_n(x)| \le C\frac{L}{\rho^2}\int_{t_{n-[\rho^2/hL]-1}}^{t_n}\int_{V(x)} r^{k-m}|\frac{\partial^k}{\partial r^k}[g(\frac{r}{\delta})u_h(t,r,\theta)]|dydt$$
$$+C\int_{t_{n-[\rho^2/hL]-1}}^{t_n}\int_{V(x)} r^{k-m}|\frac{\partial^k}{\partial r^k}[g(\frac{r}{\delta})\bar{\partial}_t u_h(t,r,\theta)]|dydt.$$

Noting the relation $kp > m$ and $(t_{n-[\rho^2/hL]-1}, t_n] \times V(x) \subset Q_\rho$, we have

$$|u_n(x)| \le C\frac{L}{\rho^2}(\int_{t_{n-[\rho^2/hL]-1}}^{t_n}\int_{V(x)} r^{(k-m)p/(p-1)}dydt)^{(p-1)/p}(\iint_{Q_\rho}|\frac{\partial^k}{\partial r^k}[g(\frac{r}{\delta})u_h(t,y)]|^p dydt)^{1/p}$$
$$+C(\int_{t_{n-[\rho^2/hL]-1}}^{t_n}\int_{V(x)} r^{(k-m)p/(p-1)}dydt)^{(p-1)/p}(\iint_{Q_\rho}|\frac{\partial^k}{\partial r^k}[g(\frac{r}{\delta})\bar{\partial}_t u_h(t,r,\theta)]|^p dydt)^{1/p}.$$

Hence from the calculation:

$$\int_{t_{n-[\rho^2/hL]-1}}^{t_n}\int_{V(x)} r^{(k-m)p/(p-1)}dydt = \int_{t_{n-[\rho^2/hL]-1}}^{t_n}(\int_\alpha\int_0^\delta r^{(k-m)p/(p-1)}r^{m-1}drd\theta)dt$$
$$= \rho^2 C(\alpha)\frac{p-1}{pk-m}\delta^{(pk-m)/(p-1)},$$

we arrive at the estimate

$$|u_n(x)| \le C(\rho)(\iint_{Q_\rho}|\frac{\partial^k}{\partial r^k}[g(\frac{r}{\delta})u_h(y,t)]|^p dydt)^{1/p}$$
$$+\widetilde{C}(\rho)(\iint_{Q_\rho}|\frac{\partial^k}{\partial r^k}[g(\frac{r}{\delta})\bar{\partial}_t u_h(r,\theta,t)]|^p dydt)^{1/p}.$$

Case2. In this case we have that $(t_n, t_{n+[\rho^2/hL]+1}] \times V(x) \subset Q_\rho(t_{n_0}, x_0)$ and we define a function $\sigma(t)$ on $(t_{n-1}, t_{n+[\rho^2/hL]+1}]$ as follows:

$$\sigma(t) = \sigma_i \quad \text{for} \quad t_{i-1} < t \le t_i,$$
$$\sigma_n = 1,$$
$$\sigma_i = -hL/\rho^2 + \sigma_{i-1} \quad \text{for} \quad n+1 \le i \le n+[\rho^2/hL],$$
$$\sigma_{n+[\rho^2/hL]+1} = 0.$$

Noting the equality

$$
\begin{aligned}
-u_n(x) &= \sum_{i=n+1}^{n+[\rho^2/hL]+1} (\sigma_i u_i(x) - \sigma_{i-1}u_{i-1}(x)) \\
&= \sum_{i=n+1}^{n+[\rho^2/hL]+1} (\sigma_i - \sigma_{i-1})u_i(x) + \sum_{i=n+1}^{n+[\rho^2/hL]+1} \sigma_{i-1}(u_i(x) - u_{i-1}(x)),
\end{aligned}
$$

we have the assertion of this case similarly as in Case 1. Hence the assertion of Lemma 2.1 has been shown.

Proof of Lemma 2.2. Let r and R be positive numbers arbitrarily given and fixed satisfying $r < R$. Integrating the inequality

$$
(4.3) \qquad |\bar{u}_R(t_n,x) - \bar{u}_r(t_n,x)|^p \le 2^{p-1}|u(t,y) - \bar{u}_R(t_n,x)|^p + 2^{p-1}|u(t,y) - \bar{u}_r(t_n,x)|^p
$$

with respect to (t,y) on $Q_r(t_n,x)$, we obtain

$$
|Q_r|\,|\bar{u}_R(t_n,x) - \bar{u}_r(t_n,x)|^p
$$

$$
(4.4) \qquad \le 2^{p-1}\iint_{Q_R(t_n,x)} |u(t,y) - \bar{u}_R(t_n,x)|^p\,dydt + 2^{p-1}\iint_{Q_r(t_n,x)} |u(t,y) - \bar{u}_r(t_n,x)|^p\,dydt\,.
$$

By virtue of the estimate (2.2), we infer from (4.4) that

$$
(4.5) \qquad |\bar{u}_R(t_n,x) - \bar{u}_r(t_n,x)| \le CR^{(m+2+p\alpha)/p}r^{-(m+2)/p}\,.
$$

Now we shall show $\{\bar{u}_R(t_n,x)\}$ is a Cauchy filter as R tends to zero. Let R be a fixed positive number and set $R_i = 2^{-i}R \ (i = 1,2,\dots)$. Then we obtain from the estimate (4.) with R and r replaced by R_i and R_{i+1} respectively that

$$
(4.6) \qquad |\bar{u}_{R_i}(t_n,x) - \bar{u}_{R_{i+1}}(t_n,x)| \le C2^{(m+2)/p}2^{-i\alpha}R^\alpha
$$

holds for each $i \ (i = 1,2,\dots)$. Summing the inequality (4.6) with respect to i from j to $k-1$, we infer that

$$
(4.7) \qquad |\bar{u}_{R_j}(t_n,x) - \bar{u}_{R_k}(t_n,x)| \le C2^{(m+2)/2}R^\alpha \sum_{i=j}^{k-1} 2^{-i\alpha} \le C\frac{2^\alpha}{2^\alpha - 1}2^{(m+2)/2}R_j^\alpha = CR_j{}^\alpha\,.
$$

Therefore, for each fixed $(t_n,x) \in$, $\{\bar{u}_{R_j}(t_n,x)\}$ $(j = 1,2,\dots)$ is a Cauchy sequence and hence there exists a unique $\widetilde{u}(t_n,x)$ such that

$$
\widetilde{u}(t_n,x) = \lim_{j\to\infty} \bar{u}_{R_j}(t_n,x)\,.
$$

Next we show that $\widetilde{u}(t_n,x)$ is independent of the choice of R. Let r be a positive number $r < R$ and put

$$
\widetilde{\widetilde{u}}(t_n,x) = \lim_{i\to\infty} \bar{u}_{r_i}(t_n,x)\,,
$$

where $r_i = 2^{-i}r \quad (i = 1, 2, \dots)$. We proceed to the estimate as follows:

(4.8) $|\tilde{\tilde{u}}(t_n, x) - \tilde{u}(t_n, x)| \leq |\tilde{\tilde{u}}(t_n, x) - \bar{u}_{r_i}(t_n, x)| + |\bar{u}_{r_i}(t_n, x) - \bar{u}_{R_i}(t_n, x)| + |\bar{u}_{R_i}(t_n, x) - \tilde{u}(t_n, x)|$.

Since $r < R$, for each integer i we can choose an integer k such that $k \geq i$ and $R_{k+1} < r_i \leq R_k$. In the inequality

$$|\bar{u}_{r_i}(t_n, x) - \bar{u}_{R_i}(t_n, x)| \leq |\bar{u}_{r_i}(t_n, x) - \bar{u}_{R_k}(t_n, x)| + |\bar{u}_{R_k}(t_n, x) - \bar{u}_{R_i}(t_n, x)|,$$

we use the inequalities (4.5) and (4.7), so that

(4.9) $$|\bar{u}_{r_i}(t_n, x) - \bar{u}_{R_i}(t_n, x)| \leq CR_k^{(m+2+p\alpha)/p} r_i^{-(m+2)/p} + CR_i^\alpha \leq C(2^{(m+2)/p} + 1)R_i^\alpha.$$

Hence, combining (4.9) with (4.8) and tending i to infinity, we have

$$\tilde{u}(t_n, x) = \tilde{\tilde{u}}(t_n, x).$$

Also, taking $j = 0$ in (4.7), we infer

$$|\bar{u}_R(t_n, x) - \bar{u}_{R_k}(t_n, x)| \leq CR^\alpha.$$

Here, tending k to infinity, we obtain

(4.10) $$|\bar{u}_R(t_n, x) - \tilde{u}(t_n, x)| \leq CR^\alpha.$$

Noting that (4.10) holds for sufficiently small $R > 0$ and that for $R, 0 < R < \sqrt{h}$,

$$\bar{u}_R(t_n, x) = \frac{1}{|B_R|} \int_{B_R(x)} u_n(y)\, dy,$$

we have

$$\lim_{R \to +0} \frac{1}{|B_R|} \int_{B_R(x)} u_n(y)\, dy = \tilde{u}(t_n, x)$$

uniformly for each $(t_n, x) \in Q$. On the other hand, since we have for each Lebesgue point $x \in \Omega$ of $u_n(\cdot), 1 \leq n \leq N$, that

$$\lim_{R \to +0} \frac{1}{|B_R|} \int_{B_R(x)} u_n(y)\, dy = u_n(x),$$

we obtain for

$$\tilde{u}(t_n, x) = u_n(x) \quad \textit{for almost all } x \in \Omega.$$

Hence, taking (4.10) into account, we arrive at the estimate

(4.11) $$|\bar{u}_R(t_n, x) - u(t_n, x)| \leq CR^\alpha \quad \text{for any} \quad x \in \Omega$$

and for any $R > 0$.

We shall claim that the assertion of Lemma 2.2 now follows from the above estimate (4.11). Let (t_n, x) and $(t_{n'}, x')$ be points in Q satisfying $\delta((t_n, x), (t_{n'}, x')) \leq \frac{1}{2}$ min $(dist(x, \partial\Omega), dist(x', \partial\Omega), \sqrt{t_n}, \sqrt{t_{n'}})$, and put

$$r = \delta((t_n, x), (t_{n'}, x'))$$

In the inequality
(4.12)
$|u(t_n,x)-u(t_{n'},x')| \le |u(t_n,x)-\bar{u}_{2r}(t_n,x)| + |\bar{u}_{2r}(t_n,x)-\bar{u}_{2r}(t_{n'},x')| + |\bar{u}_{2r}(t_{n'},x')-u(t_{n'},x')|$,

we have the estimate (4.11) for the first and third terms in the right-hand of (4.12). We shall estimate the second term . By integrating the inequality

$$|\bar{u}_{2r}(t_n,x)-\bar{u}_{2r}(t_{n'},x')| \le |\bar{u}_{2r}(t_n,x)-u(t,y)| + |u(t,y)-\bar{u}_{2r}(t_{n'},x')|$$

with respect to (t,y) over $Q_{2r}(t_n,x)\cap Q_{2r}(t_{n'},x')$, we infer

$$|Q_{2r}(t_n,x)\cap Q_{2r}(t_{n'},x')||\bar{u}_{2r}(t_n,x)-\bar{u}_{2r}(t_{n'},x')| \tag{4.13}$$

$$\le \iint_{Q_{2r}(t_n,x)} |\bar{u}_{2r}(t_n,x)-u(t,y)|\,dydt + \iint_{Q_{2r}(t_{n'},x')} |u(t,y)-\bar{u}_{2r}(t_{n'},x')|\,dydt.$$

By using Hölder inequality and (2.2), (4.13) yields

$$|\bar{u}_{2r}(t_n,x)-\bar{u}_{2r}(t_{n'},x')| \le C|Q_{2r}(t_n,x)\cap Q_{2r}(t_{n'},x')|^{-1}|Q_{2r}|^{(p-1)/p}(2r)^{(m+2+p\alpha)/p}. \tag{4.14}$$

Noticing $Q_{2r}(t_n,x)\cap Q_{2r}(t_{n'},x') \supset Q_r(t_n,x)$, we reduce (4.14) to

$$|\bar{u}_{2r}(t_n,x)-\bar{u}_{2r}(t_{n'},x')| \le Cr^{\alpha}. \tag{4.15}$$

Combining (4.11) and (4.15) with (4.12), we arrive at

$$|u(t_n,x)-u(t_{n'},x')| \le Cr^{\alpha}$$

for each (t_n,x) and $(t_{n'},x') \in Q$ satisfying $\delta((t_n,x),(t_{n'},x') \le \frac{1}{2}\min(dist(x,\partial\Omega), dist(x',\partial\Omega), \sqrt{t_n}, \sqrt{t_{n'}})$. Thus the proof of Lemma2.2 is completed.

We shall next give the proof of Lemma 2.3. For the following we assume that the condition in Lemma 2.3 is satisfied.

Let $\sigma(x)$, $|\sigma(x)|\le 1$, $|D\sigma(x)| \le 2/r$, be a smooth function belonging to $C_0^\infty(B_2)$ such that for a positive γ

$$\int_{B_2} \sigma\,dx \ge \gamma$$

and we put for a positive r

$$\sigma_r(x) = \sigma(\frac{x}{r}).$$

For the following we fix r and we rewrite σ_r by σ and hence we remark that we have the estimate

$$\int_{B_{2r}} \sigma\,dx \ge \gamma r^m \qquad \text{and} \qquad |D\sigma(x)| \le 2/r.$$

For $u_n \in L^1(\Omega, R^N)$, we define $u^\sigma_{n,r}$ by

$$u^\sigma_{n,r} = \int_{B_{2r}} u_n\sigma\,dx \Big/ \int_{B_{2r}} \sigma\,dx.$$

Proposition 4.2. *For σ defined above there exists a positive number C such that we have*

$$|u^\sigma_{n,r} - u^\sigma_{n',r}|^2 \le Cr^{-m} \int_{([0,T]\times supp\sigma)\cap Q_r(t_{n_0},x_0)} |Du|^2\, dz \tag{4.16}$$

for any r and for any positive integers n, n', $1 \le n, n' \le N$, satisfying $n > n'$ and $n_0 \ge n, n' \ge n_0 - [r^2/h]$.

Proof. Testing the identity (1.3) with a function $h(u^\sigma_{n,r} - u^\sigma_{n',r})\sigma$, we obtain for $1 \le k \le N$

$$0 = \int_{B_r} \sigma(u_k - u_{k-1})(u^\sigma_{n,r} - u^\sigma_{n',r})dx + h\int_{B_r} A^{\alpha\beta}_{ij} D_\beta u^j_k D_\alpha \sigma(u^\sigma_{n,r} - u^\sigma_{n',r})^i dx.$$

Summing the resultant equations over k from $n'+1$ to n, we infer

$$0 = \sum_{k=n'+1}^{n} \int_{B_r} \sigma(u_k - u_{k-1})(u^\sigma_{n,r} - u^\sigma_{n',r})dx + h\sum_{k=n'+1}^{n} \int_{B_r} A^{\alpha\beta}_{ij} D_\beta u^j_k D_\alpha \sigma(u^\sigma_{n,r} - u^\sigma_{n',r})^i dx.$$

We proceed to the estimate as follows:

$$\begin{aligned} &\sum_{k=n'+1}^{n} \int_{B_r} \sigma(u_k - u_{k-1})(u^\sigma_{n,r} - u^\sigma_{n',r})dx \\ &= (\int_{B_r} \sigma u_n\, dx - \int_{B_r} \sigma u_{n'}\, dx)(u^\sigma_{n,r} - u^\sigma_{n',r}) \\ &= \int_{B_r} \sigma\, dx |u^\sigma_{n,r} - u^\sigma_{n',r}|^2 \ge \gamma r^m |u^\sigma_{n,r} - u^\sigma_{n',r}|^2 \end{aligned}$$

and

$$|\sum_{k=n'+1}^{n} h\int_{B_r} A^{\alpha\beta}_{ij} D_\beta u^j_k D_\alpha \sigma(u^\sigma_{n,r} - u^\sigma_{n',r})^i dx| \le \int_{t'_n}^{t_n}\int_{B_r} |A||D\sigma||Du| dx dt |u^\sigma_{n,r} - u^\sigma_{n',r}|, \tag{4.17}$$

where $|A|$ is the operator norm of $\{A^{\alpha\beta}_{ij}\}$. Since $|D\sigma| \le 2r^{-1}$ and the assumption implies $[t_{n'}, t_n] \subset [t_{n_0} - r^2, t_{n_0}]$, it follows from (4.17) that

$$\begin{aligned} &|\sum_{k=n'+1}^{n} h\int_{B_r} A^{\alpha\beta}_{ij} D_\beta u^j_k D_\alpha \sigma(u^\sigma_{n,r} - u^\sigma_{n',r})^i dx| \\ &\le 4\varepsilon r^{-2} \int_{t_{n_0}-r^2}^{t_{n_0}} \int_{B_r} |u^\sigma_{n,r} - u^\sigma_{n',r}|^2\, dx dt + 4^{-1}\epsilon^{-1}|A|^2 \int_{([0,T]\times supp\sigma)\cap Q_r(t_{n_0},x)} |Du|^2\, dz \\ &= 4\varepsilon r^m \kappa_m |u^\sigma_{n,r} - u^\sigma_{n',r}|^2 + 4^{-1}\varepsilon^{-1}|A|^2 \int_{([0,T]\times supp\sigma)\cap Q_r(t_{n_0},x)} |Du|^2\, dz. \end{aligned}$$

Hence, by taking $\varepsilon = \gamma/8\kappa_m$, we obtain the assertion of Proposition:

$$|u^\sigma_{n,r} - u^\sigma_{n',r}|^2 \le 4\gamma^{-2}\kappa_m |A|^2 r^{-m} \int_{([0,T]\times supp\sigma)\cap Q_r(t_{n_0},x)} |Du|^2\, dz.$$

By using this property, we shall prove Lemma 2.3. We here use the notation:

$$u_r^\sigma(t) = \int_{B_r\times\{t\}} u(t,x)\sigma(x)dx / \int_{B_r}\sigma(x)dx,$$
$$u_r^\sigma = \iint_{Q_r} u(t,x)\sigma(x)dxdt / \iint_{Q_r}\sigma(x)dxdt,$$
$$u_{j,r} = \frac{1}{|B_r|}\int_{B_r} u_j(x)dx$$
$$\bar{u}_r = \bar{u}_r(t_{n_0}, x_0),$$

where $\bar{u}_r(t_{n_0}, x_0)$ is the function defined in (1.8).

Proof of Lemma 2.3. At first we shall treat ~~with~~ the case $h < r^2$. Let $B_r = B_r(x_0)$ and $Q_r = Q_r(t_{n_0}, x_0) \subset Q, 1 \le n_0 \le N, x_0 \in \Omega$, be fixed. Noting the integral $\int_{Q_r}|u-c|^2\,dz$ has the minimum when $c = \bar{u}_r$, we have

$$\int_{Q_r}|u(t,y)-\bar{u}_r|^2\,dydt \le \int_{Q_r}|u(t,y)-u_r^\sigma|^2\,dydt$$
$$\le 2\iint_{Q_r}|u(t,y)-u_r^\sigma(t)|^2\,dydt + 2\iint_{Q_r}|u_r^\sigma(t)-u_r^\sigma|^2\,dydt. \tag{4.18}$$

The first term of the right-hand side of (4.18) is calculated as follows:

$$\iint_{Q_r}|u(t,y)-u_r^\sigma(t)|^2\,dydt$$
$$= \sum_{j=n_0-[r^2/h]+1}^{n_0} h\int_{B_r}|u_j(y)-u_{j,r}^\sigma|^2\,dy + (r^2-[r^2/h]h)\int_{B_r}|u_{n_0-[r^2/h]}(y)-u_{n_0-[r^2/h],r}^\sigma|^2\,dy. \tag{4.19}$$

We shall estimate the term $|u_j(y)-u_{j,r}^\sigma|$. Noting the calculation

$$u_j(y)-u_{j,r}^\sigma = \int_{B_r}(u_j(y)-u_j(\tilde{y}))\sigma(\tilde{y})\,d\tilde{y}\Big/\int_{B_r}\sigma(x)\,dx,$$

and using Schwarz inequality, we infer

$$\int_{B_r}|u_j(y)-u_{j,r}^\sigma|^2\,dy \le \int_{B_r}\Big(\int_{B_r}|u_j(y)-u_j(\tilde{y})|\sigma(\tilde{y})\,d\tilde{y}\Big)^2\,dy\Big/\Big(\int_{B_r}\sigma(y)\,dy\Big)^2$$
$$\le \int_{B_r}\int_{B_r}|u_j(y)-u_j(\tilde{y})|^2\,d\tilde{y}\,dy\int_{B_r}\sigma(\tilde{y})^2\,d\tilde{y}\Big/\Big(\int_{B_r}\sigma(y)\,dy\Big)^2. \tag{4.20}$$

Using usual Poincarè inequality, we proceed the estimate as follows:

$$\int_{B_r}\int_{B_r}|u_j(y)-u_j(\tilde{y})|^2\,d\tilde{y}\,dy$$
$$\le 2\int_{B_r}\int_{B_r}|u_j(y)-u_{j,r}|^2\,d\tilde{y}\,dy + 2\int_{B_r}\int_{B_r}|u_j(\tilde{y})-u_{j,r}|^2\,d\tilde{y}\,dy$$
$$\le 2Cr^2\int_{B_r}\int_{B_r}|Du_j(y)|^2\,dy\,d\tilde{y} + 2Cr^2\int_{B_r}\int_{B_r}|Du_j(\tilde{y})|^2\,d\tilde{y}\,dy$$
$$= 4Cr^2|B_r|\int_{B_r}|Du_j(y)|^2\,dy. \tag{4.21}$$

Hence, by virtue of (4.20) and (4.21), we infer

$$
\begin{aligned}
&\int_{B_r} |u_j(y) - u^\sigma_{j,r}|^2\, dy \\
&\le 4Cr^2|B_r| \int_{B_r} |Du_j(y)|^2\, dy \int_{B_r} \sigma(\widetilde{y})^2\, d\widetilde{y} \Big/ \Big(\int_{B_r} \sigma(y)\, dy\Big)^2 \qquad (4.22) \\
&\le Cr^2 \int_{B_r} |Du_j(y)|^2\, dy.
\end{aligned}
$$

Consequently, combining (4.19) with (4.22), we arrive at

$$
\iint_{Q_r} |u(t,y) - u^\sigma_r(t)|^2\, dy\, dt \le Cr^2 \int_{Q_r} |Du(z)|^2\, dz. \qquad (4.23)
$$

On the other hand, the second term of the right-hand side of (4.18) is estimated as follows:

$$
\begin{aligned}
&\iint_{Q_r} |u^\sigma_r(t) - u^\sigma_r|^2\, dy\, dt \\
&= \sum_{j=n_0-[r^2/h]+1}^{n_0} h \int_{B_r} |u^\sigma_{j,r} - u^\sigma_r|^2\, dy + (r^2 - [r^2/h]h) \int_{B_r} |u^\sigma_{n_0-[r^2/h],r} - u^\sigma_r|^2\, dy \qquad (4.24) \\
&= h|B_r| \sum_{j=n_0-[r^2/h]+1}^{\cdot} |u^\sigma_{j,r} - u^\sigma_r|^2 + (r^2 - [r^2/h]h)|B_r||u^\sigma_{n_0-[r^2/h],r} - u^\sigma_r|^2.
\end{aligned}
$$

We here notice the following calculation:

$$
\begin{aligned}
u^\sigma_r &= \iint_{Q_r} u(s,\widetilde{y})\sigma_r(\widetilde{y})\, d\widetilde{y}\, ds \Big/ \iint_{Q_r} \sigma_r(\widetilde{y}) d\widetilde{y}\, ds \\
&= \int_{t_{n_0}-r^2}^{t_{n_0}} \Big(\int_{B_r} u(s,\widetilde{y})\sigma_r(\widetilde{y})\, d\widetilde{y}\Big)\, ds \Big/ \int_{B_r} \sigma_r(\widetilde{y}) d\widetilde{y} \int_{t_{n_0}-r^2}^{t_{n_0}} 1\, ds = \int_{t_{n_0}-r^2}^{t_{n_0}} u^\sigma_r(s)\, ds \Big/ \int_{t_{n_0}-r^2}^{t_{n_0}} 1\, ds \\
&= \int_{t_{n_0}-r^2}^{t_{n_0}} \int_{B_r} u^\sigma_r(s)\, d\widetilde{\widetilde{y}}\, ds \Big/ \int_{t_{n_0}-r^2}^{t_{n_0}} \int_{B_r} 1\, d\widetilde{\widetilde{y}}\, ds = \iint_{Q_r} u^\sigma_r(s)\, d\widetilde{\widetilde{y}}\, ds \Big/ \int_{Q_r} 1\, dz.
\end{aligned}
$$

Hence we infer

$$
u^\sigma_{j,r} - u^\sigma_r = \iint_{Q_r} (u^\sigma_{j,r} - u^\sigma_r(s))\, d\widetilde{\widetilde{y}}\, ds \Big/ \int_{Q_r} 1\, dz.
$$

By using Schwarz inequality, we obtain

$$
|u^\sigma_{j,r} - u^\sigma_r|^2 \le \int_{Q_r} 1\, dz \iint_{Q_r} |u^\sigma_{j,r} - u^\sigma_r(s)|^2\, d\widetilde{\widetilde{y}}\, ds \Big/ \Big(\int_{Q_r} 1\, dz\Big)^2 \qquad (4.25)
$$

$$
= |B_r|\Big(\sum_{i=\ -[r^2/h]+1}^{\cdot} h|u^\sigma_{j,r} - u^\sigma_{i,r}|^2 + (r^2 - [r^2/h]h)|u^\sigma_{j,r} - \quad {}_{-[r^2/h],r}|^2\Big) \Big/ \int_{Q_r} 1\, dz.
$$

We here use Proposition 4.2 to have

$$
|u^\sigma_{j,r} - u^\sigma_r|^2 \le Cr^{-m} \int_{Q_r} |Du|^2\, dz \qquad (4.26)
$$

for $n_0 - [r^2/h] \le j \le n_0$. Hence, combining (4.26) with (4.24), it follows

$$\int_{Q_r} |u_r^\sigma(t) - u_r^\sigma|^2\, dz \le Cr^2 \iint_{Q_r} |Du|^2\, dxdt. \tag{4.27}$$

Substituting (4.23) and (4.27) into (4.18), we obtain the assertion with the restriction $h < r^2$.

Next we treat the case $r^2 \le h$. In this case we have

$$u(t, y) = u_{n_0}(y) \qquad \text{for} \qquad t_{n_0} - r^2 < t \le t_{n_0},$$

so that

$$\bar{u}_r(t_{n_0}, x_0) = \frac{1}{|B_r|} \int_{B_r(x_0)} u_{n_0}(\tilde{y}) d\tilde{y}.$$

Hence we obtain that

$$\begin{aligned} \int_{Q_r(t_{n_0}, x_0)} |u(t,y) - \bar{u}_r(t_{n_0}, x_0)|^2\, dz &= r^2 \int_{B_r(x_0)} |u_{n_0}(y) - \frac{1}{|B_r|} \int_{B_r(x_0)} u_{n_0}(\tilde{y}) d\tilde{y}|^2 dy \\ \le Cr^4 \int_{B_r(x_0)} |Du_{n_0}(y)|^2 dy &= Cr^2 \int_{Q_r(t_{n_0}, x_0)} |Du(z)|^2 dz. \end{aligned}$$

Therefore we have proved Lemma 2.3.

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