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Critical Values of the Yamabe Functional

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§1 Introduction.

Let M be a compact connected n-dimensional smooth manifold, and C a conformal class of Riemannian metrics of M, i.e., a set of Riemannian metrics which are conformally related to one anoter. To be more precise, C is a set which can be written as

$$C = \{e^u g;\ u \in C^\infty(M)\},$$

where g is an arbitrarily fixed element of C. The Yamabe functional $I: C \to \mathbb{R}$ is then defined as

$$I(g) = \frac{\int_M R_g dv_g}{(\int_M dv_g)^{\frac{n-2}{n}}} \quad \text{for } g \in C,$$

where R_g is the scalar curvature function of g, and dv_g is the canonical volume element. The exponent (n-2)/n in the denominator is chosen so that I is invariant by scaling, i.e., I(g) = I(cg) for any positive $c \in \mathbb{R}$ and $g \in C$. It follows from the Gauss-Bonnet formula that I is constant equal to $4\pi\chi(M)$ when n = 2. In general, I is bounded below, and we denote by $\mu(M,C)$ the greatest lower bound of $I: C \to \mathbb{R}$;

$$\mu(M,C) = \inf_{g \in C} I(g).$$

The first variational formula is computed as

$$\frac{d}{dt}I(e^{tu}g)\Big|_{t=0} = \frac{n-2}{n}\frac{\int_M (R_g - r_g)u\,dv_g}{(\int_M dv_g)^{\frac{n-2}{n}}},$$

$$r_g = \int R_g dv_g / \int dv_g$$

for $u \in C^\infty(M)$ and $g \in C$. Therefore, when $n \geq 3$, $g \in C$ is a critial point of $I:C \to \mathbb{R}$ if and only if R_g = const., and $r \in \mathbb{R}$ is a critical value of $I:C \to \mathbb{R}$ if and only if there exists a metric $g \in C$ such that Vol(M,g) = 1 and R_g = const. = r. The Yamabe theorem (cf. [2,5]) asserts that $\mu(M,C)$ is in fact a critical value of $I:C \to \mathbb{R}$, and we always have at least one critical value of $I:C \to \mathbb{R}$. The purpose of this paper is to report some results concerning the following

Conjecture. The number of critical values of the Yamabe functional is finite.

§ 2 Known results.

As was mentioned, $\mu(M,C)$ is the unique critical value of I if n = 2, and we are interested in the case when $n \geq 3$. Then C can be written as

$$C = \{u^{\frac{4}{n-2}} g;\ u \in C^\infty(M) \text{ and } u > 0\},$$

for any fixed $g \in C$. It is straightforward to see

$$I(u^{\frac{4}{n-2}}g) = \frac{\frac{4(n-1)}{n-2}\int_M |du|^2 dv_g + \int_M R_g dv_g}{\left(\int_M u^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}},$$

and

$$\mu(M,C) = \inf_{u>0} I(u^{\frac{4}{n-2}}g).$$

From this expression, we can see $\mu(M,C) \leqq \max R_g$ for any $g \in C$ with $\mathrm{Vol}(M,g) = 1$. Moreover, from the Sobolev inequality, we have $\min R_g \leqq 0$ for any $g \in C$ if $\mu(M,C) \leqq 0$. Then it is easy to see the following.

Lemma. *If $\mu(M,C) \leqq 0$, then $\min R_g \leqq \mu(M,C) \leqq \max R_g$ for any $g \in C$ with $\mathrm{Vol}(M,g) = 1$.*

As a corollary we have

Fact 1. *If $\mu(M,C) \leqq 0$, $\mu(M,C)$ only is a critical value of $I: C \to \mathbb{R}$.*

Here, the assumption $\mu(M,C) \leqq 0$ is essential, and the problem becomes more difficult when $\mu(M,C) > 0$. The following fact due to M. Obata [4] is one of few results about the critical values of the Yamabe functional which are applicable to cases when $\mu(M,C) > 0$.

Fact 2. *If C contains an Einstein metric, $\mu(M,C)$ is the unique critical value of $I: C \to \mathbb{R}$.*

Now suppose $M = S^1 \times S^{n-1}$, $n \geqq 3$, and $g_\ell = ds^2 + h_0$, where h_0 is the standard metric of the Euclidean $(n-1)$-sphere S^{n-1} and ds^2 is a metric of S^1 with $\mathrm{length}(S^1, ds^2) = \ell$. Let C_ℓ be the conformal class containing the metric g_ℓ of $S^1 \times S^{n-1}$. We can apply a theorem of Gidas, Ni and Nirenberg [1] in order to get all critical points of $I:C_\ell \to \mathbb{R}$. As a result, we have

Fact 3. There are only finitely many critical values of $I:C_\ell \to \mathbb{R}$.

An interesting observation about this example is that the number of critical values increases to infinity as ℓ goes to infinity.

§ 3 Almost critical values.

So far we have seen a few results which are partial evidences to our conjecture. Here we change slightly our point of view.

Definition. $r \in \mathbb{R}$ is called an almost critical value of the Yamabe functional $I:C \to \mathbb{R}$ if for any $\varepsilon > 0$ there is a metric $g \in C$ with $Vol(M,g) = 1$ such that $|R_g(x) - r| < \varepsilon$ for all $x \in M$.

It is clear that every critical value is almost critical.

Theorem. If $\mu(M,C) > 0$ and $\dim M \geq 3$, there are infinitely many almost critical values of the Yamabe functional $I:C \to \mathbb{R}$. Otherwise, $\mu(M,C)$ only is an almost critical value of I.

Proof. The latter part follows immediately from the lemma in § 2. The first part is a restatement of Theorem 4 of [3]. □

References.

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