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Level sets of the solution to elliptic boundary value problems

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§ 1. Introduction

Convexity of level sets of solutions to elliptic boundary value problems over convex domain in $\mathbb{R}^n$ ($n \geq 2$) was shown by many authors (see Kawohl [10] and its references). For example, let us consider the unique solution u to the Saint Venant torsion problem : " $\Delta u = -1$ in Ω and $u = 0$ on $\partial\Omega$ ", where Ω is a bounded convex domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. In the case $n = 2$, Makar-Limanov [16] first showed that all the level sets of the solution u , $\{ x \in \Omega ; u(x) \geq t \}$ are strictly convex. Furthermore, in the general case $n \geq 2$, Kennington [12] and Kawohl [11] showed that $\sqrt{u}$ is concave and the level sets are convex. Also, Kennington [12] pointed out that the strict concavity of $\sqrt{u}$ had been proved in Makar-Limanov [16] in the case $n = 2$. The strict concavity of $\sqrt{u}$ was shown by Korevaar & Lewis [15] in the general case. Thus the convexity of level sets was obtained as a consequence of the concavity of $\sqrt{u}$. There are similar facts for solutins to the other elliptic boundary value problems over convex domain in $\mathbb{R}^n$ (see [1, 2, 3, 4, 6, 8, 9, 11, 12, 13, 14,.15]) Namely, the convexity of level sets was obtained as a consequence of the concavity of $g(u)$ for some monotone function $g(\cdot)$. Another well known example is the fact that the logarithm of the first positive eigenfunction of the Laplacian on a convex domain in $\mathbb{R}^n$ is concave (see [1, 2, 3, 4, 8, 14]).

On the other hand, in the two-dimensional problems the uniqueness of critical point of solutions was shown by several authors for some broader classes of nonlinear elliptic boundary value problems, though the convexity of level sets is not known. Of course the convexity of level sets implies the uniqueness of critical point. Concerning the Dirichlet problems, we know the results in Sperb [20] for semilinear elliptic equations and the results in Philippin [17] for quasilinear ones. Concerning the problems which are not Dirichlet, we know the results of Chen [5] for the capillary surfaces(see also [18] and see [19] for semilinear elliptic equations). In view of these results, for example we conjecture that positive solutions to the semilinear elliptic problem over bounded convex domain Ω in $\mathbb{R}^n$: $\Delta u = f(u)$ in Ω and $u = 0$ on $\partial\Omega$ have only convex level sets.

In this note we give a new proof of Makar-Limanov's theorem. Precisely, we prove the convexity of level sets without showing the concavity of the root of the solution in the two-dimensional case. There may be a little hope to consider the above conjecture. Finally in this section we state Makar-Limanov's theorem.

Theorem (Makar-Limanov [16]). Let Ω be a bounded convex domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a unique solution to the Dirichlet problem

(1) $$\Delta u = -1 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

Then all the level sets of u are strictly convex. That is, the curvature of level curves does not vanish.

§ 2. A new proof of Makar-Limanov's theorem.

It suffices to show

Lemma 8.1. Let u be a unique solution to (1). If $\xi \cdot \nabla u(p) = 0$ for a point $p \in \Omega$ and a direction $\xi \in \mathbb{R}^2$ ($|\xi| = 1$), then the second derivative of u with respect to the direction ξ at p does not vanish, that is, $\Sigma\, \xi_i \xi_j D_{ij} u(p) \neq 0$.

Proof. By using a parallel transformation and a rotation of coordinates, we may assume that $\xi = (1,0)$ and $p = 0$. Suppose that $D_1 u(0) = D_{11} u(0) = 0$. Since u is real analytic, by Taylor's formula we get

(2) $$u(x) = w(x) + O(|x|^3) \quad \text{as} \quad x \to 0,$$

where $w(x) = a + bx_2 + cx_1x_2 - \frac{1}{2}x_2^2$ for some numbers a, b, c. Here we used the equation that $\Delta u(0) = -1$. Moreover, observing that $\Delta w = -1$ and $\Delta(u-w) = 0$ in Ω, we see that for some integer $n \geq 3$

(3) $$(u-w)(x) = P_n(x) + O(|x|^{n+1}) \quad \text{as } x \to 0,$$

where $P_n(x)$ is a harmonic homogeneous polynomial of degree n and $P_n(x)$ is not identically zero. On the other hand, it follows from the result of Hartman & Wintner [7, Corollary 1, p.

450] that every interior critical point of $u-w$ is isolated. Therefore as in [6] we see that the zero set of $u-w$ in some neighborhood U of origin consists of n smooth arcs, all intersecting at origin and dividing U into $2n$ sectors ($n \geq 3$). Put $A = \{ x \in \Omega;\ u(x) - w(x) > 0 \}$ and $B = \{ x \in \Omega;\ u(x) - w(x) < 0 \}$. Then, it follows from the maximum principle that

(4) <u>Both</u> A <u>and</u> B <u>have at least three components each of which meets the boundary</u> $\partial\Omega$.

Now consider the zero set of w. Since $a = u(0) > 0$, we see that for $c \neq 0$ it is a hyperbolic curve, one of whose asymptotic lines is x_1- axis, and for $c = 0$ it consists of two parallel lines. Let $\Omega^{\sim}$ be the connected component of $\{ x \in \Omega;\ w(x) > 0 \}$ containing the origin. Then, since Ω is convex, the boundary of $\Omega^{\sim}$ consists of at most four connected arcs, each of which belongs to $\partial\Omega$ or the zero set of w alternatively. Therefore, there are at most only two components of $\{ x \in \Omega^{\sim};\ u(x) > w(x) \}$, each of which meets the boundary $\partial\Omega^{\sim}$. This contradicts the fact (4). This completes the proof.

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