数理解析研究所講究録
第738巻 1991年 73-78

Applications of Lyusternik–Schnirelmann theory to Hamiltonian systems

by K. Hayashi (Meiji Univ.)
林喜代司

§1 Introduction

Let $x = (x^1, x^2, \ldots, x^n)$ and $p = (p_1, p_2, \ldots, p_n)$ be points of $\mathbf{R}^n$ and consider a Hamiltonian system

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}; \quad i = 1, 2, \ldots, n\ , \tag{1.1}$$

where $H = H(x,p) : \mathbf{R}^n \to \mathbf{R}$ is a C^∞ function (Hamiltonian function) and $\dot{}$ means $\frac{d}{dt}$. Along a solution $(x(t), p(t))$ of (1.1), $H(x(t), p(t))$ is a constant, so, for given e, the *energy surface* $H^{-1}(e) \equiv \{(x,p); H(x,p) = e\}$ is an invariant set. If $H^{-1}(e)$ is not compact, then there is not necessarily a periodic solution on it.

On the existence of periodic solutions of Hamiltonian systems on energy surface, P. Rabinowitz [6] obtained a remarkable

Theorem 1. *If $H^{-1}(e)$ is star shaped, then there exist at least one periodic solution of (1.1) on it.*

For this theorem, the Hamiltonian function $H(x,p)$ is an arbitrary function. But originary in tha classical mechanics, the Hamiltonian function had a special form, namely "*kinetic energy + potential*". This means $H(x,p)$ is of the form

$$H(x,p) = \frac{1}{2} a^{ij}(x) p_i p_j + U(x), \tag{1.2}$$

where (a^{ij}) is symmetric and positive definite. We call the Hamiltonian system (1.1) with Hamiltonian function of the form (1.2) a *classical* Hamiltonian system. Then we have [1] [2]

Theorem 2. *For classical Hamiltonian systems, if $H^{-1}(e)$ is compact, then there exists at least one periodic solution on it.*

In order to obtain more than one periodic solutions on compact energy surfaces of classical Hamiltonian systems, we have an eye to the following point. We put $T = \frac{1}{2} a^{ij}(x) p_i p_j$, then we have $T \geq 0$. Hence, if a point (x,p) satisfies $T + U = e$, then $U(x) \leq e$. Thus we consider, for a fixed e, the set

(1.3) $$W \equiv \{x; U(x) \leq e\}.$$

Remark that "$H^{-1}(e)$ is compact" if and only if "W is compact".

From now on, we assume that e is a regular value of H (equivalently of U). Then W is a compact manifold with boundary $[U = e]$. In this note, we propose a conjecture " there may be at least $\nu(W)$ periodic solutions on the energy surface of the classical Hamiltonian system", and give some circumstantial evidence of this conjecture. The number $\nu(W)$ is a topological invariant of W given below.

§2 Geodesics as solutions of (1.1)

For a classical Hamiltonian (1.2), the Hamiltonian system (1.1) is equivalent to the Lagrangian system

(2.1) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i}, \quad i = 1, 2, \ldots, n,$$

where $L = T - U$ is the Lagrangian with

(2.2) $$T = T(x, \dot{x}) \equiv \frac{1}{2}a_{ij}(x)\dot{x}^i\dot{x}^j, \qquad (a_{ij}) = (a^{ij})^{-1}.$$

If $(x(t), p(t))$ is a solution of (1.1), (1.2) on $H^{-1}(e)$, then $x(t)$ is a solution of (2.1) with $T + U = e$. Conversely, if $x(t)$ is a solution of (2.1), then $T(x, \dot{x}) + U(x)$ is a constant e and $(x(t), p(t))$ is a solution of (1.1), (1.2) on $H^{-1}(e)$, where $p(t)$ is properly determined by $x(t)$. Also, it is known [Maupeutuis–Jacobi's variational principle] that the above $x(t)$ is, after a time change, a geodesic for a Riemannian metric

(2.3) $$ds^2 = (e - U(x))\frac{1}{2}a_{ij}(x)dx^i dx^j.$$

This metric is called *Jacobi metric* for e. This Jacobi metric is positive on $\mathrm{Int}\, W = [U < e]$ and degenerate on $\partial W = [U = e]$. Maupertuis–Jacobi's principle gives

Lemma 1 *If $\gamma : [0,1] \to W$ is a C^∞ curve with*

- *$\gamma(s)$ is a geodesic for the Jacobi metric in* $\mathrm{Int}\, W$,
- *$\gamma(0), \gamma(1) \in \partial W$,*

then $(x(t), p(t))$, *where* $x(t)$ *is obtained by* $\gamma(s)$ *after proper time change* $t \leftrightarrow s$ *and* $p(t)$ *is determined from* $x(t)$ *as above, is a periodic solution of* (1.1) *on* $H^{-1}(e)$.

In fact, let $x(t)$ be the solution of (2.1) with

- $x(t)$ in $\operatorname{Int} W$, $\quad t_0 < t < t_1$,
- $x(t_0), x(t_1) \in \partial W$,

for some $t_0 < t_1$. Then the solution $x(t)$ stops at the times $t = t_0$ and t_1, because on the boundary $[U = e]$, we have $T = e - U = 0$ at the times, hence $\dot{x} = 0$. By the reversibility of the system (2.1), the inverse curve $x(t_1 - t)$ is also a solution of (2.1) with same total energy $T + U$. This stops again at $t = t_1 + (t_1 - t_0)$. Connecting these solutions

- $x(t), \quad t_0 \le t \le t_1$,
- $x(t_1 - t), \quad t_1 \le t \le t_1 + (t_1 - t_0)$,
- $x(t), \quad t_1 + (t_1 - t_0) \le t \le t_1 + 2(t_1 - t_0)$,
- $\cdots$,

we have a desired periodic solution.

As pointed out above, the Jacobi metric is degenerate on $\partial W = [U = e]$. To avoid this, we consider a compact manifold W_δ, which is contained in $\operatorname{Int} W$ and diffeomorphic to W, as follows.

Fix a small $\delta > 0$. For $b \in B = \partial W$, let $x_b(t)$ be the solution of (2.1) with $x_b(0) = b$, $\dot{x}(0) = 0$, and $t(b, \delta)$ the first time for which the length of the curve $x_b(t), 0 \le t \le t(b, \delta)$, with respect to the Jacobi metric equals to δ. We put

$$b_\delta = x_b(t(b, \delta)) \quad \text{and} \quad B_\delta = \bigcup_{b \in B} b_\delta.$$

Finally let W_δ be the compact set consisting of the points "inside" B_δ. For sufficiently small δ, $W_\delta \approx W$ and it is known that if a geodesic with respect to Jacobi metric intersect with B_δ orthogonally, then the geodesic can be extended so as to reach the boundary B. We call a geodesic of a compact manifold with boundary an *orthogonal geodesic chord*, if it starts and ends at points of the boundary orthogonally. The above consideration and Lemma 1 give

Lemma 2 *Orthogonal geodesic chords of W_δ with respect to the Jacobi metric give periodic solutions of the original Hamiltonian system* (1.1) *with* (1.2) *on* $H^{-1}(e)$.

§3 Lyusternik–Schnirelmann theory for orthogonal geodesic chords

For the existence and the number of orthogonal geodesic chords of compact Riemannian manifolds with boundary, the following is known.

Theorem 3 *Let Y be a compact Riemannian manifold with geodesically convex boundary. Then we have at least $\nu(Y)$ orthogonal geodesic chords.*

The topological invariant $\nu(Y)$ is defined as follows. We put $B = \partial Y \neq \emptyset$ and

$$\Omega_Y \equiv \{\omega : [0,1] \to Y; \text{continuous and } \omega(0), \omega(1) \in B\} \tag{3.1}$$

with compact open topology. In the following, the coefficients of the (co)homology shall be understood as $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$. We define

1. $\nu_\pi(Y) = \begin{cases} 1 & \text{if } \pi_k(\Omega_Y, B) \neq 0 \text{ for some } k \geq 1, \\ 0 & \text{otherwise} \end{cases}$
2. if $H_*(\Omega_Y, B) = 0$, then $\nu_H(Y) = 0$ and otherwise
$$\begin{aligned}\nu_H(Y) = \text{Max}\ \{k \geq 1\ ;\ & \exists \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \in H^*(\Omega_Y) \text{ with } \deg \alpha_j > 0 \\ & \text{and } \exists\ a\ \in H_*(\Omega_Y, B) \\ & \text{such that } \quad (\alpha_1 \cup \cdots \cup \alpha_{k-1}) \cap a \neq 0 \quad \}\end{aligned}$$
3. $\nu_\Pi(Y)$ is obtained as $\nu_H(Y)$, exchanging $H^*(\Omega_Y)$ and $H_*(\Omega_Y, B)$ to $H^*_\Pi(\Omega_Y)$ and $H^\Pi_*(\Omega_Y, B)$. Here, H^*_Π and H^Π_* are equivariant (co)homology with respect to the involution $\omega \mapsto \omega^{-1} \equiv \omega(1 - \cdot)$.
4. $\nu(Y) \equiv \text{Max}\{\nu_\pi(Y), \nu_H(Y), \nu_\Pi(Y)\}$.

The proof is given by Lyusternik–Schnirelmann theory applied to the following variational problem. Let Λ be the path space consisting of all piecewise C^∞ pathes $\lambda : [0,1] \to Y$ with $\lambda(0), \lambda(1) \in B$. Also define $E : \Lambda \to \mathbf{R}$ by

$$E(\lambda) = \frac{1}{2}\int_0^1 dt |\dot{\lambda}(t)|^2. \tag{3.2}$$

Nontrivial ($E > 0$) "critical points" of E correspond to nonconstant orthogonal geodesic chords. The assumption of the geodesical convexity corresponds to the condition (C) of

Palais–Smale. For example, let $a \in H_k(\Lambda, B)$ be a nonzero element (remark that Λ is homotopically equivalent to Ω_Y). For a representative of a

$$z = \sum_i \sigma_i, \quad \sigma_i : \Delta^k \to \Lambda, \text{ singular simplex,}$$

we put

$$|z| = \bigcup_i \operatorname{Im} \sigma_i \subset \Lambda$$

and define

$$\kappa_a \equiv \inf_{z \in a} \operatorname{Max} E(|z|).$$

Then κ_a is a nontrivial critical value.

If there is an $\alpha \in H^*(\Lambda)$ with $\deg \alpha > 0$ satisfying $b \equiv \alpha \cap a \neq 0$, then, in general, $\kappa_b \leq \kappa_a$ and when $\kappa_b = \kappa_a$, there exist infinitely many critical points on the level. This means, in that case, there exist at least two critical points, giving the meaning of the definition of $\nu_H(Y)$.

The topological invariant $\nu(Y)$ has the following properties.

1. for any Y, we have $\nu(Y) \geq 1$.
2. if Y is contractible, then $\nu(Y) = \dim Y$, in particular $\nu(D^n) = n$.
3. for solid torus $S^1 \times D^2$, we have $\nu(S^1 \times D^2) \geq 3$.

Corresponding to these properties, we have the following results on the classical Hamiltonian systems.

1. there is always at least one periodic orbit [1] [2].
2. when $W \approx D^n$, there exist at least n periodic solutions for systems near a rotationally symmetric one [3].
3. when $W \approx S^1 \times D^2$, there exist at least 3 periodic solutions for systems near one with some symmetry [5].

Thus it is plausible that the following may be valid: *on a compact energy surface of a classical Hamiltonian system, there may be at least* $\nu(W)$ *periodic solutions on it.*

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