数理解析研究所講究録
第 738 巻 1991 年 91-107

On H_0^1-estimates for radially symmetric solutions of semilinear elliptic equations

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1. Introduction. We consider the radially symmetric solutions for the semilinear elliptic equation

(1.1) $$\Delta u + g(u) = 0, \qquad x \in \Omega,$$

(1.2) $$u = 0, \qquad x \in \partial\Omega,$$

where $\Omega \equiv \{ x \in \mathrm{R}^n : |x| < 1 \}$, $n \geq 2$, and $g(s)$ is a continuous function such that

(g_1) $$g(0) = 0, \quad \lim_{|s|\to\infty} g(s)/s = \infty \quad \text{and}$$
$$g'(0) = \lim_{s\to 0} g(s)/s \quad \text{exists.}$$

The equation for radially symmetric solutions $u = u(t)$, $t = |x|$, is the following form.

(1.3) $$u'' + \frac{n-1}{t} u' + g(u) = 0, \qquad t \in (0, 1),$$

(1.4) $$u'(0) = 0, \qquad u(1) = 0.$$

Under some conditions on $g(s)$, Ambrosetti and Rabinowitz [1] established that for any bounded domain Ω the problem (1.1)-(1.2) possesses infinitely many solutions and moreover $H_0^1(\Omega)$ norms of solutions assume arbitrarily large values. The related problems are treated in [2, 5, 8, 10, 12, 13, 14]. In the case where Ω is the unit ball, the existence of infinitely many radially symmetric solutions has been investigated by Castro-Kurepa [4] and Struwe [15] (see [3] for $n = 1$). In fact, Struwe [15] has proved by a variational method that there is an integer k_0 such that for any $k \geq k_0$ the problem (1.3)-(1.4) admits a solution with exactly k zeros in [0, 1]. On the other hand, using a shooting method, Castro and Kurepa [4] have showed the same results under weaker assumptions on $g(s)$. Therefore we will study the relation between $H_0^1(\Omega)$ norms of radially symmetric solutions and the numbers of their zeros.

2. Main results. Recall that Ω is the unit ball in $\mathbf{R}^n$. We denote by $L^r(\Omega)$ ($1 \leq r \leq \infty$) and by $H_0^1(\Omega)$ the usual Lebesgue and Sobolev spaces, respectively. The norm of $L^r(\Omega)$ is denoted by $\|\cdot\|_r$. The $H_0^1(\Omega)$ norm is defined by

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_2 = \left(\int_\Omega |\nabla u(x)|^2 \, dx\right)^{1/2}.$$

Let H be the subspace of $H_0^1(\Omega)$ which consists of radially symmetric functions. We define the norm $\|\cdot\|_H$ of H by

$$\|u\|_H = \omega_n^{-1/2} \|u\|_{H_0^1(\Omega)} = \left(\int_0^1 u'(t)^2 t^{n-1} \, dt\right)^{1/2},$$

where ω_n means the surface area of the unit sphere $\partial\Omega$. We write S for the set of all solutions $u \in C^2(0, 1) \cap C^1[0, 1]$ of (1.3)-(1.4). For $k \in \mathrm{N}$, S_k denotes the set of all solutions which have exactly k zeros in the interval [0, 1]. Let n^* be defined by $n^* = \infty$ if $n = 2$ and $n^* = \frac{n+2}{n-2}$ if $n \geq 3$. We state our first result which gives the lower bounds of solutions.

Theorem 1 (lower estimates) *Suppose the condition* (g_1) *and there are constants* $p \in (1, n^*)$ *and* $a_i > 0$ *such that*

(g_2) $\quad sg(s) \le a_1|s|^{p+1} + a_2 \quad$ *for all* $s \in \mathbf{R}$.

Then there are constants $c_1, c_2 > 0$ *such that*

$$c_1 k^{(p+1)/(p-1)} - c_2 \le \|u\|_H \qquad \text{for any } u \in S_k \text{ and } k \ge 1.$$

Theorem 1 may assert nothing for small k in the case of $c_1 < c_2$. However, as stated below, we see that any nontrivial solution is bounded away from the trivial solution.

Theorem 2. *In addition to conditions* (g_1) *and* (g_2) *suppose that* $g'(0)$ *is not an eigenvalue of the problem*:

(2.1) $$-v'' - \frac{n-1}{t} v' = \lambda v, \qquad t \in (0, 1),$$

(2.2) $$v'(0) = v(1) = 0.$$

Then there is a constant $c > 0$ *such that*

$$c \le \|u\|_H \qquad \text{for all } u \in S \setminus \{0\}.$$

The result stated in Theorem 1 is optimal, since we also have upper estimates for the solutions.

Theorem 3 (upper estimates). *Set* $G(s) = \int_0^s g(r)\,dr$. *In addition to* (g_1) *suppose the following two conditions*:

(g_3) $$\limsup_{|s|\to\infty} \frac{sg(s)}{G(s)} < n^* + 1.$$

There are constants $p \in (1, n^*)$, $b_i > 0$ *and* $R > 0$ *such that*

(g_4) $\qquad b_1|s|^{p+1} \le sg(s) \le b_2|s|^{p+1} \qquad$ *for all* $|s| \ge R$.

Then there is a constant $c > 0$ *such that*

$$\|u\|_H \le ck^{(p+1)/(p-1)} \qquad \text{for any } u \in S_k \text{ and } k \ge 1.$$

Finally, we consider the Emden-Fowler equation which involves a typical nonlinear term $g(s)$.

Example (Emden-Fowler Equation). Consider the equation

$$u'' + \frac{n-1}{t}u' + |u|^{p-1}u = 0, \qquad t \in (0, 1), \tag{2.3}$$

$$u'(0) = u(1) = 0, \tag{2.4}$$

where $p \in (1, n^*)$. We shall show that for each $k \ge 1$ the problem (2.3)-(2.4) possesses a unique solution which has exactly k zeros in [0, 1] and satisfies $u(0) > 0$. If we denote the solution by $u_k(t)$, then it follows that

$$S_k = \{u_k, -u_k\} \quad \text{and} \quad S = \{0\} \cup \{\pm u_k : k \in \mathrm{N}\}.$$

To prove these assertions, we consider (2.3) on $[0, \infty)$ together with the initial condition,

$$u'(0) = 0 \text{ and } u(0) = 1. \tag{2.5}$$

It is easy to verify that equation (2.3) with (2.5) possesses a unique global solution $w(t)$ on $[0, \infty)$. Furthermore the solution $w(t)$ is oscillatory (see [11, Corollary 6.7]). Here $w(t)$ is said to be oscillatory if $w(t)$ has an unbounded sequence of zeros in $[0, \infty)$. Since $w(t)$ has at most finite zeros in any bounded interval, we may denote the set of all zeros of $w(t)$ as $\{s_k\}_{k=1}^{\infty}$ $(0 < s_1 < s_2 < \cdots \uparrow \infty)$. First we observe that $\lambda^{2/(p-1)}w(\lambda t)$ $(\lambda > 0)$, as well as $w(t)$, satisfies equation (2.3) on $[0,\infty)$. Set $u_k(t) \equiv s_k^{2/(p-1)} w(s_k t)$. Then the function $u_k(t)$ is a solution of (2.3)-(2.4) which has exactly k zeros in [0, 1] and satisfies $u_k(0) > 0$. Such a solution is unique, since any solution of (2.3) satisfying $u'(0) = 0$ and $u(0) > 0$ can be written in the form $\lambda^{2/(p-1)} w(\lambda t)$ $(\lambda > 0)$. In view of these facts we see that $S_k = \{ u_k, \; - u_k\}$ and $S = \{0\} \cup \{ \pm u_k : k \in \mathrm{N} \}$. We now apply Theorems 1 through 3 to find the asymptotic distribution in $H_0^1(\Omega)$ of solutions of (2.3)-(2.4). That is, there exist constants c_1, $c_2 > 0$ such that

$$c_1 k^{(p+1)/(p-1)} \le \|u_k\|_H \le c_2 k^{(p+1)/(p-1)} \qquad \text{for any} \quad k \in \mathrm{N}.$$

3. Proofs of Theorems 1 and 2

Proof of Theorem 2. We prove Theorem 2 by contradiction. Suppose that there is a sequence of nontrivial solutions $\{u_j\}_{j=1}^{\infty} \subset S \setminus \{0\}$ such that $\lim_{j\to\infty} \|u_j\|_H = 0$. Then using " Moser's iteration technique " (cf. [6, 9]), we have $\lim_{j\to\infty} \|u_j\|_\infty = 0$. Set $v_j(x) = \|u_j\|_\infty^{-1} u_j(x)$; each v_j satisfies

$$- \Delta v_j = (g(u_j)/u_j)v_j \qquad \text{for} \quad x \in \Omega.$$

Since the right hand side is bounded in $L^\infty(\Omega)$, $\{v_j\}_{j=1}^{\infty}$ is bounded in $W^{2,q}(\Omega)$ and is relatively compact in $W^{1,q}(\Omega)$ for any $q \in [1, \infty)$. Hence one finds a subsequence of $\{v_j\}_{j=1}^{\infty}$ which converges to some function $v(x)$ in both $W^{1,q}(\Omega)$ and $L^\infty(\Omega)$. This together with $\|v_j\|_\infty = 1$ implies

$$- \Delta v = g'(0)v, \quad (x \in \Omega); \qquad v = 0, \; (x \in \partial\Omega); \qquad \|v\|_\infty = 1.$$

Since $v(x)$ is radially symmetric, $g'(0)$ becomes an eigenvalue of (2.1)-(2.2). This contradicts the assumptions of Theorem 2 and the proof is complete. □

To prove Theorem 1 we need the following two lemmas.

Lemma 1. *For each* $u \in S$, *we have*

$$\int_0^1 u'(t)^2 t^{n-1}\, dt = \int_0^1 u(t) g(u(t)) t^{n-1}\, dt. \tag{3.1}$$

Proof. Multiplying (1.3) by $u(t)t^{n-1}$ and applying integration by parts, we obtain the desired relation (3.1). □

The next lemma is obtained by applying Sturm's comparison theorem.

Lemma 2 ([7, p346, Corollary 5.2]). *Let* $q(t)$ *be a continuous function on* $[a, b]$. *Let* $v(t) \not\equiv 0$ *be a solution of the equation*:

$$v'' + q(t)v = 0, \qquad t \in [a, b].$$

Assume that $v(t)$ *has exactly* k *zeros in* $(a, b]$. *Then*

$$k < \frac{1}{2}\left((b-a)\int_a^b q^+(t)\, dt\right)^{1/2} + 1,$$

where $q^+(t) \equiv \max\{ q(t), 0 \}$.

Proof of Theorem 1. We shall prove Theorem 1 in the simple case only where $g(s) = |s|^{p-1}s$, since the same methods are valid for the general case of $g(s)$ as well (see [9]). That is, we consider the Emden-Fowler equation (2.3)-(2.4). Let $u \in S_k$. In what follows, we denote various constants independent of u and k by C (> 0). In the case of $n \geq 3$, we employ the following Liouville transformation:

$$r = t^{1/\alpha}, \qquad v(r) = r^{\beta}u(t), \qquad 2\beta = (n-2)\alpha + 1,$$

where α (> 1) is a constant to be determined later. Then (2.3) is reduced to

$$v''(r) + q(r)v(r) = 0, \qquad r \in (0, 1],$$

where $q(r) \equiv \alpha^2 r^{2\alpha-2}|u(r^\alpha)|^{p-1} - \beta(\beta-1)r^{-2}$.

Since $q(r)$ and $v(r)$ satisfy the assumptions of Lemma 2 on the interval $[\varepsilon, 1]$ for sufficiently small $\varepsilon > 0$, it follows that

$$k < \frac{1}{2}\{(1-\varepsilon)\int_\varepsilon^1 q^+(r)\, dr\}^{1/2} + 1 \le \frac{1}{2}\{\int_0^1 q^+(r)\, dr\}^{1/2} + 1$$
$$\le C\{\int_0^1 |u(r^\alpha)|^{p-1}r^{2\alpha-2}\, dr\}^{1/2} + 1.$$

From Hölder's inequality,

$$(3.2) \qquad k \le C\{\int_0^1 |u(t)|^{p+1}t^{n-1}\, dt\}^{(p-1)/2(p+1)}$$
$$\times\{\int_0^1 t^{\gamma(p+1)/2}\, dt\}^{1/(p+1)} + C,$$

where $\gamma \equiv 1 - 1/\alpha - (n-1)(p-1)/(p+1)$. Since $p \in (1, n^*)$, one can choose $\alpha > 1$ so large that $\gamma(p+1)/2 > -1$. Then the function $t^{\gamma(p+1)/2}$ is integrable over $[0, 1]$. From (3.2) and (3.1) we obtain the desired estimate for $n \ge 3$. In the case of $n = 2$, we employ the Liouville transformation

$$r = \frac{1}{1 - \log t}, \qquad v(r) = ru(t),$$

to reduce (2.3) to the equation,

$$v'' + q(r)v = 0, \qquad r \in (0, 1],$$

where $q(r) \equiv t^2|\log(t/e)|^4|u(t)|^{p-1}$ with $t = e^{(r-1)/r}$. Applying Lemma 2 and using Hölder's inequality, we obtain the assertion for $n = 2$ in the same way as in the case of $n \geq 3$. The proof is complete. □

4. Proof of Theorem 3 The purpose of this section is to prove Theorem 3. Here we deal with the only case where $g(s) = |s|^{p-1}s$. However, our methods are applicable to the nonlinear term $g(s)$ satisfying the assumptions of Theorem 3 (see [9]). In order to prove Theorem 3 we need several lemmas.

Lemma 3. *There are constants* $\theta > 0$ *and* $c_i > 0$ $(1 \leq i \leq 3)$ *such that*

$$\max_{0\leq t\leq 1} |u(t)|^{p+1}t^n \leq c_1\int_0^1 |u(t)|^{p+1}t^{n-1}\,dt$$

$$\leq c_2\int_0^1 |u(t)|^{p+1}t^{2n-3+\theta}\,dt \leq c_3\max_{0\leq t\leq 1} |u(t)|^{p+1}t^n$$

for any $u \in S$.

Proof. Multiplying (2.3) by $u'(t)t^m$ $(m > 0)$ and integrating over $[0, T]$, we have

$$\frac{1}{2}\, u'(T)^2T^m + \frac{1}{p+1}|u(T)|^{p+1}T^m = \frac{m}{p+1}\int_0^T |u(t)|^{p+1}t^{m-1}\,dt + \left(\frac{m}{2}+1-n\right)\int_0^T u'(t)^2t^{m-1}\,dt. \tag{4.1}$$

First we substitute $m = n$ into (4.1) to obtain

$$|u(T)|^{p+1}T^n \leq n\int_0^T |u(t)|^{p+1}t^{n-1}\,dt \leq n\int_0^1 |u(t)|^{p+1}t^{n-1}\,dt$$

for all $T \in [0, 1]$, which implies the first inequality of Lemma

3. Secondly, the substitution of $m = n$ and $T = 1$ into (4.1) yields

$$\frac{1}{2}u'(1)^2 = \left(\frac{n}{p+1} - \frac{n-2}{2}\right)\int_0^1 |u(t)|^{p+1}t^{n-1}\,dt, \tag{4.2}$$

where we have used (3.1). Thirdly, substituting $m = 2n-2+\theta$ and $T = 1$ into (4.1), we have

$$\begin{aligned}\frac{1}{2}u'(1)^2 &= \frac{2n-2+\theta}{p+1}\int_0^1 |u(t)|^{p+1}\,t^{2n-3+\theta}\,dt \\ &\quad + \frac{\theta}{2}\int_0^1 u'(t)^2\,t^{2n-3+\theta}\,dt \\ &\le \frac{2n-2+\theta}{p+1}\int_0^1 |u(t)|^{p+1}\,t^{2n-3+\theta}\,dt \\ &\quad + \frac{\theta}{2}\int_0^1 |u(t)|^{p+1}\,t^{n-1}\,dt,\end{aligned} \tag{4.3}$$

where we have used (3.1) and $n-2+\theta > 0$. Combining (4.2) and (4.3), one finds

$$\begin{aligned}&\left(\frac{n}{p+1} - \frac{n-2+\theta}{2}\right)\int_0^1 |u(t)|^{p+1}\,t^{n-1}\,dt \\ &\le \frac{2n-2+\theta}{p+1}\int_0^1 |u(t)|^{p+1}\,t^{2n-3+\theta}\,dt.\end{aligned} \tag{4.4}$$

Since $p \in (1, n^*)$, we can choose $\theta > 0$ so small that $n/(p+1) - (n-2+\theta)/2 > 0$. Then (4.4) implies the second inequality of Lemma 3. Finally it is not difficult to check the last inequality of Lemma 3, and the proof is complete. □

In view of Lemma 3, we introduce the next notation for convenience.

Definition 1. We define

$$M(u) \equiv \max_{0\le t\le 1} |u(t)|^{p+1}t^n \qquad \text{for} \quad u \in S.$$

By Lemma 3 and (3.1) there is a constant $c_4 > 0$ such that

$$\int_0^1 u'(t)^2\, t^{n-1}\, dt \le c_4\, M(u) \tag{4.5}$$

for any $u \in S$. Hence, to prove Theorem 3, it is sufficient to compute the value of $M(u)$ instead of the $H_0^1(\Omega)$ norm of u.

The next lemma, which follows readily from Sturm's comparison theorem, is useful for proving Lemma 5 below.

Lemma 4 (cf. [7, p336, Exercise 3.2]). *In the differential equation*

$$(p(t)u')' + q(t)u = 0, \qquad t \in [a, b], \tag{4.6}$$

let $p(t) \in C^1[a, b]$ *and* $q(t) \in C[a, b]$ *satisfy*

$$p(t) \ge m \quad \text{and} \quad M \ge q(t),$$

where m *and* M *are positive constants. If* $u(t)$ *is a solution with a pair of zeros* t_1, t_2 ($t_1 < t_2$) *and* $u(t) \not\equiv 0$, *then we have*

$$t_2 - t_1 \ge \pi\left(\frac{m}{M}\right)^{1/2}.$$

To evaluate $M(u)$ we subdivide the interval [0, 1] in the following way:

Definition 2. Let $u \in S_k$ and $\{t_i\}_{i=1}^{k}$ ($0<t_1<t_2<\cdots<t_k=1$) denote its zeros. Let $\delta \in (0, 1/2)$. Moreover we set

$$\delta_i = \delta(t_{i+1} - t_i), \qquad \delta_0 = 2\delta t_1,$$

$$I_i = [t_i + \delta_i,\ t_{i+1} - \delta_i], \qquad 1 \le i \le k-1,$$

$$I_0 = [0,\ t_1 - \delta_0],$$

$$I = \cup_{i=0}^{k-1} I_i \quad \text{and} \quad J = [0, 1] \setminus I.$$

In what follows, we denote various constants by C_ε, C_δ, $C_{\varepsilon,\delta}$ and C; C_ε means a constant depending upon ε, $C_{\varepsilon,\delta}$ represents a constant depending possibly on ε and δ, and C denotes an absolute constant.

Lemma 5. *Let $u \in S_k$ and $\varepsilon \in (0, 1)$. Then we have*

(i) $\displaystyle\int_{I_i} |u(t)|^{p-1}\, t^{n-1}\, dt \le \frac{C_\delta}{t_{i+1} - t_i}, \qquad 1 \le i \le k-1;$

(ii) $\displaystyle\int_{I_0} |u(t)|^{p-1}\, t^{n-1}\, dt \le C_\delta;$

(iii) *if* $t_i \ge \varepsilon$, *then* $\displaystyle\frac{1}{t_{i+1} - t_i} \le C_\varepsilon\, M(u)^{(p-1)/2(p+1)}.$

Proof. Consider the eigenvalue problem

$$-(t^{n-1}v')' - a(t)v = \lambda t^{n-1}v, \qquad t \in (\alpha, \beta),$$

$$v(\alpha) = v(\beta) = 0,$$

where $\alpha = t_i$, $\beta = t_{i+1}$, $i \ge 1$ and $a(t) \equiv |u(t)|^{p-1}\, t^{n-1}$. We see that $\lambda = 0$ is the first eigenvalue. In fact, $v(t) = u(t)$ and $\lambda = 0$ satisfy the above equation and moreover $u(t)$ has no zeros in the interval $(\alpha, \beta) = (t_i, t_{i+1})$. Therefore $\lambda = 0$ is the first eigenvalue, and so we have the following variational characterization:

$$\min_{v \in V} \frac{\displaystyle\int_\alpha^\beta \{v'(t)^2 t^{n-1} - a(t)v(t)^2\}\, dt}{\displaystyle\int_\alpha^\beta v(t)^2\, t^{n-1}\, dt} = 0, \tag{4.7}$$

where V denotes the set of all functions $v \in C^1[\alpha, \beta]$ satisfying $v(\alpha) = v(\beta) = 0$ and $v \not\equiv 0$. It follows from (4.7) that

$$\int_\alpha^\beta v'(t)^2 t^{n-1}\, dt \ge \int_\alpha^\beta a(t)v(t)^2\, dt$$

$$\ge \left(\min_{t\in I_i} v(t)^2\right) \int_{I_i} a(t)\, dt$$

for any $v \in V$. We choose $v(t) \equiv (\beta - t)(t - \alpha) \in V$ to get

$$\frac{1}{3}(\beta - \alpha)^3 \ge \left(\frac{\delta^2}{4}\right)(\beta - \alpha)^4 \int_{I_i} a(t)\, dt.$$

This implies the assertion (i). Next we prove the assertion (ii). In the same argument as above we have

$$\int_0^{t_1} v'(t)^2\, t^{n-1}\, dt \ge \left(\min_{t\in I_0} v(t)^2\right) \int_{I_0} a(t)\, dt$$

for any $v \in C^1[0, 1]$ satisfying $v'(0) = v(t_1) = 0$. Substituting $v(t) \equiv t_1^2 - t^2$, we obtain the assertion (ii). We then prove the last assertion (iii). The function $u(t) \in S_k$ satisfies the differential equation (4.6) with $p(t) = t^{n-1}$ and $q(t) = |u(t)|^{p-1}t^{n-1}$. The functions $p(t)$ and $q(t)$ are estimated as

$$p(t) \ge \varepsilon^{n-1} \quad\text{and}\quad q(t) \le C_\varepsilon M(u)^{(p-1)/(p+1)} \qquad t \in [t_i, t_{i+1}],$$

where we used $t_i \ge \varepsilon$. Therefore assertion (iii) follows from Lemma 4. The proof is thereby complete. □

Lemma 6. *Let* θ (> 0) *be defined as in Lemma* 3. *Let* $\varepsilon \in (0, 1)$. *Then for* $t_j \ge \varepsilon$, *we get*

$$\Sigma_{i=j}^{k-1} \int_{I_i} |u(t)|^{p+1} t^{2n-3+\theta}\, dt \le C_{\varepsilon,\delta}\, k\, M(u)^{(p+3)/2(p+1)}.$$

Proof. Let $t_i \ge \varepsilon$. We set $\xi = \theta - 2 + n(p-1)/(p+1)$. Then Lemma 5 implies

$$\int_{I_i} |u(t)|^{p+1} t^{2n-3+\theta}\, dt$$

$$= \int_{I_i} \left(|u(t)|^{p+1} t^n\right)^{2/(p+1)} t^{\xi}\, |u(t)|^{p-1} t^{n-1}\, dt$$

$$\le \max(\varepsilon^{\xi}, 1)\, M(u)^{2/(p+1)}\, C_{\varepsilon,\delta} M(u)^{(p-1)/2(p+1)}$$

$$\le C_{\varepsilon,\delta} M(u)^{(p+3)/2(p+1)}.$$

Summing up both sides with respect to $i = j, j+1, \cdots, k-1$, we obtain the desired inequality. This completes the proof. □

Lemma 7.

$$\int_{[\varepsilon,1]\cap I} |u(t)|^{p+1} t^{2n-3+\theta}\, dt$$

$$\le C\varepsilon^{a} M(u) + C_{\varepsilon,\delta}\, k^{2(p+1)/(p-1)}$$

for any $\varepsilon \in (0, 1/2)$, *where* $a = n - 2 + \theta$ (> 0).

Proof. There are the two cases to be considered.

(A) There is an integer $i \in [1, k-1]$ such that $t_i \in [\varepsilon, 2\varepsilon]$;

(B) Any t_i $(1 \le i \le k-1)$ does not belong to $[\varepsilon, 2\varepsilon]$.

We can find an integer $j \in [1, k-1]$ such that $t_{j-1} < \varepsilon \le t_j \le 2\varepsilon$ in case (A), where we understand that $t_0 = 0$. On the other hand, there is an integer $j \in [1, k]$ such that $t_{j-1} < \varepsilon < 2\varepsilon < t_j$ in case (B). In either case, it follows from Lemma 6 that

$$(4.8) \quad \int_{[\varepsilon,1]\cap I} |u(t)|^{p+1} t^{2n-3+\theta}\, dt$$
$$\leq \int_{[\varepsilon,t_j]\cap I_{j-1}} |u(t)|^{p+1} t^{2n-3+\theta}\, dt + C_{\varepsilon,\delta}\, k\, M(u)^{(p+3)/2(p+1)}.$$

Now suppose $[\varepsilon, t_j] \cap I_{j-1} \neq \phi$. We want to estimate

$$K \equiv \int_{[\varepsilon,t_j]\cap I_{j-1}} |u(t)|^{p+1} t^{2n-3+\theta}\, dt.$$

First, in case (A) it follows from the direct computation that

$$(4.9) \quad K \leq \int_{\varepsilon}^{2\varepsilon} |u(t)|^{p+1} t^{2n-3+\theta}\, dt \leq \frac{2^a - 1}{a}\, \varepsilon^a\, M(u).$$

Secondly, in case (B), we have

$$K \leq C_\varepsilon\, M(u)^{2/(p+1)}\, K_{j-1},$$

where

$$K_{j-1} \equiv \int_{I_{j-1}} |u(t)|^{p-1}\, t^{n-1}\, dt.$$

Since $t_{j-1} < \varepsilon < 2\varepsilon < t_j$, Lemma 5 implies that

$$K_{j-1} \leq \frac{C_\delta}{t_j - t_{j-1}} \leq \varepsilon^{-1} C_\delta \qquad \text{if } j \geq 2, \text{ and}$$
$$K_{j-1} \leq C_\delta \qquad \text{if } j = 1.$$

Therefore we obtain

$$(4.10) \quad K \leq C_{\varepsilon,\delta}\, M(u)^{2/(p+1)}.$$

From (4.8), (4.9) and (4.10), we have

$$\int_{[\varepsilon,1]\cap I} |u(t)|^{p+1} t^{2n-3+\theta}\, dt$$

$$\leq C\varepsilon^a\ M(u) + C_{\varepsilon,\delta}\ k\ M(u)^{(p+3)/2(p+1)} + C_{\varepsilon,\delta}\ M(u)^{2/(p+1)}.$$

Thus, we can apply Young's inequality to the right hand side to get the conclusion. □

We now give the proof of Theorem 3.

Proof of Theorem 3. We set $a = n - 2 + \theta$ (> 0) as in Lemma 7. Let $\varepsilon, \delta \in (0, 1/2)$. In order to evaluate $M(u)$, we estimate

$$\begin{aligned} &\int_0^1 |u(t)|^{p+1}\ t^{2n-3+\theta}\ dt \\ (4.11)\qquad &= \left(\int_0^\varepsilon + \int_{[\varepsilon,1]\cap J} + \int_{[\varepsilon,1]\cap I}\right) |u(t)|^{p+1}\ t^{2n-3+\theta}\ dt. \end{aligned}$$

First, we see from Definition 2 that the measure of J is 2δ. Therefore the first and the second terms can be estimated as

$$(4.12)\qquad \int_0^\varepsilon |u(t)|^{p+1}\ t^{2n-3+\theta}\ dt \leq \frac{\varepsilon^a}{a}\ M(u)$$

and

$$(4.13)\qquad \int_{[\varepsilon,1]\cap J} |u(t)|^{p+1}\ t^{2n-3+\theta}\ dt \leq C(\delta^a + \delta)\ M(u),$$

respectively. The last term on the right hand side of (4.11) has already been estimated in Lemma 7. Using (4.11), (4.12), (4,13) and Lemmas 3 and 7, we have

$$(4.14)\qquad M(u) \leq C(\varepsilon^a + \delta^a + \delta)\ M(u) + C_{\varepsilon,\delta}\ k^{2(p+1)/(p-1)},$$

where C is independent of u, k, ε and δ. We now choose

ε, $\delta > 0$ so small that $C(\varepsilon^a+\delta^a+\delta) < 1/2$. Then (4.14) implies

$$M(u) \le C\, k^{2(p+1)/(p-1)}.$$

Thus, the desired assertion is obtained by applying (4.5). The proof is thereby complete. □

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