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A GENERALIZATION OF THE SECRETARY PROBLEM WITH UNCERTAIN EMPLOYMENT

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ABSTRACT. The secretary problem with uncertain employment, as Smith (1975) called it, is here generalized. n rankable candidates appear in random order. Only their relative ranks are observed. As each candidate appears, we must decide either to give an offer or not. When an offer is given, the candidate accepts it with a fixed probability $p(0<p\leq 1)$. We continue observations until an offer is accepted or the final candidate is interviewed. Our objective is to examine a strategy which maximizes the probability of employing the most preferred available candidate.

1. INTRODUCTION. We first review the classical secretary problem : n candidates appear one by one in random order with all n! permutations equally likely. We are able at any time to rank the candidates that have so far appeared according to some preference order. As each candidate appears we must decide either to choose or disregard that candidate with the objective of maximizing the probability of choosing the best (most preferred) candidate. It is assumed that each candidate accepts an offer of employment with certainty and that the disregarded candidate cannot be recalled later.

To make the problem more realistic, many authors have attempted to relax the assumptions imposed on the classical secretary problem. One modification is to incorporate flexibility in recalling a previously disregarded candidate. Works along this line are due to Rose (1984 a,b), Smith and Deely (1975), and Yang (1974)(see also Petruccelli 1982, and Tamaki 1986, considered in the full information version). Another modification of interest is to allow the candidate the right of refuse an offer of acceptance. Smith (1975) considered the problem in which the candidate only accepts an offer with a known fixed probability, under the criterion of maximizing the probability of choosing the best candidate.

Though Smith's problem is simple and tractable as a first step, it is unrealistic in that it seeks the overall best rather than seeking the best among those who will accept an offer. This lack of realism motivates our problem. Let us call a candidate who accepts an offer *available* , and say that our trial is a *success* if we can choose the best available candidate. As in Smitl each candidate here is assumed to be available with probability $p(0<p\leqq 1)$, independent of the rank of that candidate and the arrangement of the other candidates. Then our problem can be described as finding a strategy of maximizing the probability of success, based on both the relative ranks and the availabilities observed so far.

2. THE OPTIMAL STRATEGY. Imagine a situation where r candidate: for a position have so far appeared and unsuccessful (unacceptabl‹ offers have been made to k of them, $1\leqq k\leqq r\leqq n$ (the k=0 case is considered later). This situation is represented by $(r;i_1,i_2,...,i_k)$, where the *information pattern* $(i_1,i_2,...,i_k)$ represents the relative ranks (among the first r candidates) of these candidates who have declined offers, arranged in the ascending order, i.e., $1\leqq i_1<i_2<...<i_k\leqq r$. For example, state (4;1,3) is the situation where unsuccessful offers have been made to two candidates, who are the best and the third best among the first four candidates. Further, denote by $(r+1;i \mid i_1,i_2,...,i_k)$, $1\leqq i\leqq r+1$, a situation where, after leaving state $(r;i_1,i_2,...,i_k)$, we have just observed that the $(r+1)^{st}$ candidate is the i^{th} best among those that have so far appeared. In this situation we must decide either to make an offer or not.

Let $v_r(i_1,i_2,...,i_k)$ be the probability of success, assuming we proceed optimally after leaving state $(r;i_1,i_2,...,i_k)$. Also let $s_{r+1}(i \mid i_1,i_2,...,i_k)(c_{r+1}(i \mid i_1,i_2,...,i_k))$ be the corresponding probability when we make an offer to (when we decline to make an offer to) the current candidate in state $(r+1;i \mid i_1,i_2,\ldots,i_k)$ and proceed optimally thereafter. Since, from the same probability condition on the availability, the information pattern at hand has no infuluence on estimating the future arrival of the remaining candidates we,easily find that the transition from state $(r;i_1,i_2,\ldots,i_k)$ into state $(r+1;i \mid i_1,i_2,\ldots,i_k)$ occurs with probability $1/(r+1)$, independent of i and of the information pattern $(i_1,i_2,\ldots,i_k)$. We thus have for $1 \leqq k \leqq r$ and $1 \leqq r \leqq n-1$

(1) $$v_r(i_1,i_2,\ldots,i_k) = (r+1)^{-1} \sum_{i=1}^{r+1} \max\{ s_{r+1}(i \mid i_1,i_2,\ldots,i_k), c_{r+1}(i \mid i_1,i_2,\ldots,i_k)\} ,$$

with the boundary condition $v_n(i_1,i_2,\ldots,i_k) \equiv 0$, $1 \leqq k \leqq n$.

To derive the recurrence relation of $s_{r+1}(i \mid i_1,i_2,\ldots,i_k)$, define $g_{r+1}(i \mid i_1,i_2,\ldots,i_k)$ as the conditional probability that the $(r+1)^{st}$ candidate is in fact the most preferred available candidate, given that this candidate has accepted the offer made in state $(r+1;i \mid i_1,i_2,\ldots,i_k)$. It is easily shown that $g_{r+1}(i \mid i_1,i_2,\ldots,i_k)$ depends on the information pattern $(i_1,i_2,\ldots,i_k)$ only through the *reference number* j, the number of candidates who have received offers and who are more preferred to the current candidate ($(r+1)^{st}$ candidate). More specifically

$$j = \max\{ s, \ 0 \leqq s \leqq k : \ i_s < i \},$$

where i_0 is interpreted as 0. Hence, we can denote $g_{r+1}(i \mid i_1,i_2,\ldots,i_k)$ by $g_{r+1}(i,j)$.

If the candidate rejects the offer given in state $(r+1;i \mid i_1,i_2,\ldots,i_k)$, the information pattern $(i_1,i_2,\ldots,i_k)$ is changed by incrementing i_s by one for $s \geqq j+1$ and then adding i as the $(j+1)^{st}$ component (when $j=k$, this change is interpreted as only adding i as the $(k+1)^{st}$ component). Thus,

(2) $$s_{r+1}(i \mid i_1, i_2, \ldots, i_k) = \begin{cases} pg_{r+1}(i,0) + qv_{r+1}(i, i_1+1, i_2+1, \ldots, i_k+1) & \text{for } 1 \leqq i \leqq i_1 \text{ and } j=0 \\ pg_{r+1}(i,j) + qv_{r+1}(i_1, i_2, \ldots, i_j, i, i_{j+1}+1, \ldots, i_k+1) & \text{for } i_j < i \leqq i_{j+1} \text{ and } 1 \leqq j < k \\ pg_{r+1}(i,k) + qv_{r+1}(i_1, i_2, \ldots, i_k, i) & \text{for } i_k < i \leqq r+1 \text{ and } j=k, \end{cases}$$

where $q=1-p$.

Simiraly if we do not make an offer to the candidate, the information pattern is changed by incrementing i_s by one for $s \geqq j+1$ (when $j=k$, no change occurs in the information pattern). Thus,

(3) $$c_{r+1}(i \mid i_1, i_2, \ldots, i_k) = \begin{cases} v_{r+1}(i_1+1, i_2+1, \ldots, i_k+1) & \text{for } 1 \leqq i \leqq i_1 \text{ and } j=0 \\ v_{r+1}(i_1, i_2, \ldots, i_j, i_{j+1}+1, \ldots, i_k+1) & \text{for } i_j < i \leqq i_{j+1} \text{ and } 1 \leqq j < k \\ v_{r+1}(i_1, i_2, \ldots, i_k) & \text{for } i_k < i \leqq r+1 \text{ and } j=k. \end{cases}$$

To describe the evolution of the process completely, we must add situation $(r;\phi)$, where no offer has been made to any of the first r candidates, $1 \leqq r \leqq n$, and where ϕ is the null set. Also denote by $(r+1; i \mid \phi)$ the situation, where after leaving state $(r;\phi)$ we have observed that the $(r+1)^{st}$ candidate is the i^{th} best among those that have so far appeared, $1 \leqq i < r+1$. Associated with these situations are the optimal value functions $v_r(\phi)$, $s_{r+1}(i \mid \phi)$, and $c_{r+1}(i \mid \phi)$, which satisfy, for $0 \leqq r < n$,

(4) $$v_r(\phi) = (r+1)^{-1} \sum_{i=1}^{r+1} \max\{ s_{r+1}(i \mid \phi), c_{r+1}(i \mid \phi)\} ,$$

where

(5) $$s_{r+1}(i \mid \phi) = pg_{r+1}(i,0) + qv_{r+1}(i)$$

and

(6) $$c_{r+1}(i \mid \phi) = v_{r+1}(\phi).$$

The boundary condition is $v_n(\phi)=0$.

Once $g_r(i,j)$ for $0 \leqq j < i \leqq r$ and $1 \leqq r \leqq n$ are given, Equations (1)-(6) can be solved recursively to yield the optimal strategy and the probability of success $v^* = v_0(\phi)$.

Assume that we make an offer in state $(r; i \mid i_1, i_2, ..., i_k)$ with the reference number j and that the r^{th} candidate accepts it. Then the probability that this candidate is the ℓ^{th} best among all is calculated as

(7) $$p(\ell \mid r, i) = \binom{\ell-1}{i-1}\binom{n-\ell}{r-i}\bigg/\binom{n}{r} \qquad i \leqq \ell \leqq n-r+i,$$

which is clearly independent of the information pattern $(i_1, i_2, ..., i_k)$(e.g., see DeGroot 1970 or Lindley 1961). On the other hand, given that the (absolute) rank of the candidate is ℓ, the conditional probability that this candidate is the best available is $q^{\ell-1-j}$, since there are $\ell-1$ better candidates and only j of them have been already ascertained to be unavailable. This leads to the following lemma.

LEMMA 1.1. For $0 \leqq j < i \leqq r$ and $1 \leqq r \leqq n$,

(8) $$g_r(i,j) = \sum_{\ell=i}^{n-r+i} q^{\ell-j-1}\, p(\ell \mid r, i).$$

Though $g_r(i,j)$ can be calculated directly from this lemma, it is convenient to use the recurrence relation given in the next lemma in order to establish the monotonicity properties of $g_r(i,j)$.

LEMMA 1.2. $g_r(i,j)$ satisfies, for $0 \leqq j < i \leqq r$ and $1 \leqq r < n$,

(9) $$g_r(i,j) = (r+1)^{-1}\{\, q i g_{r+1}(i+1,j+1) + (r+1-i) g_{r+1}(i,j)\}$$

with the boundary condition $g_n(i,j) = q^{i-j-1}$ for $0 \leqq j < i \leqq n$.

Proof. Simply denote by $(r; i, j)$ the situation $(r; i \mid i_1, i_2, ..., i_k)$ with the reference number j and assume that an offer is given in (r;i,j) and the r^{th} candidate accepts it. Then this candidate can be viewed from three different aspects at time r+1 depending on the quality and the availability of the $(r+1)^{st}$ candidate. That is

1. If the $(r+1)^{st}$ candidate is less preferred to the r^{th}, then the r^{th} candidate can be regarded as chosen in (r+1;i,j).
2. If the $(r+1)^{st}$ candidate is more preferred to the r^{th} and unavailable, then the r^{th} candidate can be regarded as chosen in (r+1;i+1,j+1).
3. If the $(r+1)^{st}$ candidate is more preferred to the r^{th} and available, then the chosen candidate cannot be the most preferred candidate.

Since these three cases occur with probability (r+1-i)/(r+1), qi/(r+1), and pi/(r+1), respectively, Equation (9) follows.

It is not difficult to show (9) by direct substitution from (8).

Let, for $2 \leq r \leq n$, $A_r = \{ (i,j) : 0 \leq j \leq i-2,\ 2 \leq i \leq r \}$, and $B_r = \{ (i,j) : 0 \leq j \leq i-1,\ 1 \leq i \leq r-1 \}$, then we can prove the following lemma by exploiting Lemma 1.2.

LEMMA 1.3. For $2 \leq r \leq n$,

(i) $g_r(i,j) \leq g_r(i,j+1)$, for $(i,j) \in A_r$

(ii) $g_r(i,j) \geq g_r(i+1,j+1)$, for $(i,j) \in B_r$

(iii) $g_r(i,j) \geq g_r(i+1,j)$, for $(i,j) \in B_r$

(iv) $g_r(i,j) \geq g_{r-1}(i,j)$, for $(i,j) \in B_r$

Proof. Note that (iv) follows from (ii) since, from (9),

$$g_r(i,j) - g_{r-1}(i,j) = (i/r)\{ g_r(i,j) - qg_r(i+1,j+1)\} .$$

We prove (i)-(iii) by induction on r. For r=n, the results are straightforward from $g_n(i,j)=q^{i-j-1}$. Assume that (i)-(iv) hold for r=m+1. Then,

(i) The result is immediate from (9) and the induction hypothesis that, for each i, $g_{m+1}(i,j)$ is nondecreasing in j.

(ii) From (9)

$$g_m(i,j) - g_m(i+1,j+1)$$
$$= (m+1)^{-1}[\{ g_{m+1}(i,j) - qg_{m+1}(i+1,j+1)\}$$
$$+ (m-i)\{ g_{m+1}(i,j) - g_{m+1}(i+1,j+1)\}$$
$$+ q(i+1)\{ g_{m+1}(i+1,j+1) - g_{m+1}(i+2,j+2)\}] .$$

Thus the result follows from the induction hypothesis for (ii).

(iii) Similarly, from (9)

$$g_m(i,j) - g_m(i+1,j)$$
$$= (m+1)^{-1}[\{ g_{m+1}(i,j) - qg_{m+1}(i+1,j+1)\}$$
$$+ (m-i)\{ g_{m+1}(i,j) - g_{m+1}(i+1,j)\}$$
$$+ q(i+1)\{ g_{m+1}(i+1,j+1) - g_{m+1}(i+2,j+1)\}] .$$

Thus the result follows from the induction hypothesis for (ii) and (iii).

The following lemma is important for the characterization of the optimal strategy.

LEMMA 1.4. $v_r(i_1,i_2,...,i_k)$ is nonincreasing in each argument.

Proof. Define for $1 \leqq t \leqq k$

$$v_r^{(t)}(i_1,i_2,...,i_k) = v_r(i_1,...,i_{t-1},i_t,i_{t+1},...,i_k) - v_r(i_1,...,i_{t-1},i_t+1,i_{t+1},...,i_k),$$

when $i_{t+1}-i_t>1$ (i_{k+1} is interpreted as $r+1$).
Then we need to prove

$$(10) \qquad v_r^{(t)}(i_1,i_2,...,i_k) \geqq 0$$

for all r, k, and information pattern $(i_1,i_2,...,i_k)$.

Proof is by induction on r. For r=n, (10) is trivial. Assume that (10) holds for r=m+1. Let $i_t'=i_t+1$. It then follows from (1)

$$v_m^{(t)}(i_1,i_2,\dots,i_k)$$

$$= (m+1)^{-1} \sum_{i=1}^{m+1} [\ \max\{\ s_{m+1}(i \mid i_1,\dots,i_{t-1},i_t,i_{t+1},\dots,i_k),\ c_{m+1}(i \mid i_1,\dots,i_{t-1},i_t,i_{t+1},\dots,i_k)\} - \max\{\ s_{m+1}(i \mid i_1,\dots,i_{t-1},i_t',i_{t+1},\dots,i_k),\ c_{m+1}(i \mid i_1,\dots,i_{t-1},i_t',i_{t+1},\dots,i_k)\}]\ .$$

Hence, in order to establish (10) for r=m, it suffices to show that, for each i,

(11) $$s_{m+1}(i \mid i_1,\dots,i_{t-1},i_t,i_{t+1},\dots,i_k) - s_{m+1}(i \mid i_1,\dots,i_{t-1},i_t',i_{t+1},\dots,i_k) \geqq 0$$

and

(12) $$c_{m+1}(i \mid i_1,\dots,i_{t-1},i_t,i_{t+1},\dots,i_k) - c_{m+1}(i \mid i_1,\dots,i_{t-1},i_t',i_{t+1},\dots,i_k) \geqq 0.$$

Straightforward calculation from (2) implies that the left side of (11) is

$$\begin{cases} qv_{m+1}^{(t+1)}(i_1,\dots,i_s,i,i_{s+1}+1,\dots,i_k+1) & \text{for } i_s<i\leqq i_{s+1} \text{ and } 0\leqq s<t, \\ p\{\ g_{m+1}(i,t) - g_{m+1}(i,t-1)\} + q\{\ v_{m+1}^{(t+1)}(i_1,\dots,i_t,i,i_{t+1}+1,\dots,i_k+1) + v_{m+1}^{(t)}(i_1,\dots,i_t,i+1,i_{t+1}+1,\dots,i_k+1)\} & \text{for } i=i_t', \\ qv_{m+1}^{(t)}(i_1,\dots,i_s,i,i_{s+1}+1,\dots,i_k+1) & \text{for } i\neq i_t',\ i_s<i\leqq i_{s+1}, \text{ and } t\leqq s\leqq k. \end{cases}$$

The induction hypothesis and Lemma 1.3 (i) proves (11). Similarly, from (3), the left side of (12) becomes

$$\begin{cases} v_{m+1}^{(t)}(i_1,\ldots,i_s,i_{s+1}+1,\ldots,i_k+1) \\ \qquad\qquad\qquad \text{for } i\neq i_t',\ i_s<i\leqq i_{s+1},\ \text{and } 0\leqq s\leqq k, \\ v_{m+1}^{(t)}(i_1,\ldots i_t,i_{t+1}+1,\ldots,i_k+1) \\ \quad + v_{m+1}^{(t)}(i_1,\ldots,i_{t-1},i_t',i_{t+1}+1,\ldots,i_k+1) \\ \qquad\qquad\qquad \text{for } i=i_t'. \end{cases}$$

This, combined with the induction hypothesis, proves (12). Thus the proof is complete.

The following theorem is the main result of this paper.

THEOREM 1.5. Assume that we are in state $(r;i \mid i_1,i_2,\ldots,i_k)$. Then there exists, depending on both r and the information pattern $(i_1,i_2,\ldots,i_k)$, an integer $d_r(i_1,i_2,\ldots,i_k)$ in $\{0,1,\ldots,r\}$, such that the optimal policy is to make an offer to the r^{th} candidate if and only if $i\leqq d_r(i_1,i_2,\ldots,i_k)$, where

$$d_r(i_1,i_2,\ldots,i_k) \\ = \max\{\ i\ :\ s_r(i \mid i_1,i_2,\ldots,i_k) \geqq c_r(i \mid i_1,i_2,\ldots,i_k)\}$$

with $\max(\phi)=0$.

The equality $d_r(i_1,i_2,\ldots,i_k)=0$ corresponds to declining an offer to the r^{th} candidate, regardless of the value of i. When in state $(r;i \mid \phi)$, there exists an integer $d_r(\phi)$ such that the above statement holds with $d_r(i_1,i_2,\ldots,i_k)$ replaced by $d_r(\phi)$, where

$$d_r(\phi) = \max\{\ i\ :\ s_r(i \mid \phi) \geqq c_r(i \mid \phi)\}\ .$$

Proof. It suffices to show that $s_r(i \mid i_1,i_2,\ldots,i_k)$ is nonincreasing in i, while $c_r(i \mid i_1,i_2,\ldots,i_k)$ is nondecreasing in i.

Consider two consecutive intervals $(i_{t-1},i_t]$ and $(i_t,i_{t+1}]$. It follows from (2), Lemma 1.3 (iii), and Lemma 1.4 that $s_r(i \mid i_1,i_2,\ldots,i_k)$ is nonincreasing on each interval. Moreover we see from (2), and Lemma 1.3 (ii) that

$$s_r(i_t \mid i_1,i_2,\ldots,i_k) - s_r(i_t+1 \mid i_1,i_2,\ldots,i_k)$$
$$= p\{ g_r(i_t,t-1) - g_r(i_t+1,t)\}$$
$$\geqq 0.$$

Thus $s_r(i \mid i_1,i_2,\ldots,i_k)$ is nonincreasing.

On the other hand, (3) shows that $c_r(i \mid i_1,i_2,\ldots,i_k)$ is constant on each interval. Also we have, as an immediate consequence of Lemma 1.4,

$$c_r(i_t+1 \mid i_1,i_2,\ldots,i_k) - c_r(i_t \mid i_1,i_2,\ldots,i_k)$$
$$= v_r^{(t)}(i_1,\ldots,i_t,i_{t+1}+1,\ldots,i_k+1)$$
$$\geqq 0.$$

Thus $c_r(i \mid i_1,i_2,\ldots,i_k)$ is nondecreasing in i. The same reasoning applies to (5)-(6) in a similar manner.

Table I presents, for given n and p, the numerical values of $d_r(i_1,i_2,\ldots,i_k)$ for each r and the possible information pattern $(i_1,i_2,\ldots,i_k)$. Evidently, $d_n(i_1,i_2,\ldots,i_k)\equiv n$. Take, for exampl the case n=4 and p=0.5. Since $d_1(\phi)=1$, we make an offer to the 1^{st} candidate and terminate the trial if this candidate accepts the offer. Otherwise, we continue by observing the 2^{nd} candidate Thus, in the latter case, we must decide whether to make an offer to the 2^{nd} candidate based on $d_2(1)$, not on $d_2(\phi)$. That is, only $d_2(1)$ serves as the basis for decision making, which is why $d_2(\phi)$ is excluded from Table I .

Table II gives v^* as a function of n and p, and Figure 1 illustrates the behavior of v^*. This figure suggests that, as n goes to infinity, v^* approaches some limit, which is decreasing i p.

It is of interest to investigate whether optimal policies contain threshold policies : a policy is said to be rank-(time-) isotone if there exists number $d_r(i_1,i_2,\ldots,i_k)(r^*(i;i_1,i_2,\ldots,i_k$ such that the optimal decision in $(r;i \mid i_1,i_2,\ldots,i_k)$ is to give an offer if and only if $i\leqq d_r(i_1,i_2,\ldots,i_k)(r\geqq r^*(i;i_1,i_2,\ldots,i_k))$. Theorem 1.5 shows that there exists a rank-isotone optimal policy

Table I

Effective decision numbers for some values of n (the 1st integer correponds to p=0.5, and the 2nd corresponds to p=0.9)

Decision number	n = 2	3	4	5	6	7	8
$d_1(\phi)$	1 1	1 0	1 0	0 0	0 0	0 0	0 0
$d_2(\phi)$		– 1	– 1	1 1	1 0	1 0	1 0
$d_2(1)$		2 –	2 –	– –	– –	– –	– –
$d_3(\phi)$			– 1	2 1	1 1	1 1	1 1
$d_3(1)$			– 2	2 1	2 –	1 –	1 –
$d_3(1,2)$			3 –	– –	– –	– –	– –
$d_4(\phi)$				3 1	2 1	1 1	1 1
$d_4(1)$				3 2	2 1	2 1	2 1
$d_4(2)$				3 –	– –	– –	– –
$d_4(1,2)$				3 2	3 –	2 –	2 –
$d_5(\phi)$					3 1	2 1	2 1
$d_5(1)$					3 2	2 2	2 1
$d_5(2)$					3 –	– –	– –
$d_5(1,2)$					4 3	3 2	2 1
$d_5(1,2,3)$					4 –	3 –	3 –
$d_6(\phi)$						3 1	2 1
$d_6(1)$						3 2	3 2
$d_6(2)$						4 –	3 –
$d_6(1,2)$						4 3	3 2
$d_6(1,2,3)$						5 3	4 2
$d_6(1,2,3,4)$						5 –	4 –
$d_7(\phi)$							3 1
$d_7(1)$							4 2
$d_7(2)$							4 –
$d_7(1,2)$							4 3
$d_7(1,3)$							4 –
$d_7(2,3)$							5 –
$d_7(1,2,3)$							5 4
$d_7(1,2,3,4)$							5 4
$d_7(1,2,3,4,5)$							6 –

We failed to prove time-isotonicity of the policy given in Theorem 1.5 but Table I suggests that this policy satisfies time-isotonicity since $d_r(i_1, i_2, \ldots, i_k)$ is nondecreasing in r.

It is difficult to analyze the properties of the decision number $d_r(i_1, i_2, \ldots, i_k)$. Table I suggests that the decision number is nondecreasing in each argument of the information pattern among the class of the effective decision numbers. However, this is not true among the class of all decision numbers ; e.g., $d_3(1)=2$, $d_3(2)=1$ for $n=4$ and $p=0.9$. We present several conjectures concerning the decision number in light of our computational experience (Table I) :

1. Assume that it is optimal to make an offer to the r^{th} candidate and that the offer is not accepted. Then it is also optimal to make an offer to the $(r+1)^{st}$ candidate if this candidate is more preferred to the r^{th}.
2. The number $d_r(i_1, i_2, \ldots, i_k)$ satisfies the inequality

$$\begin{aligned} & d_r(i_1, i_2, \ldots, i_k) \\ \leq\ & d_r(i_1, \ldots, i_s, i_a, i_{s+1}, \ldots, i_k) \qquad \text{if } i_s < i_a < i_{s+1}. \end{aligned}$$

3. The number $d_r(i_1, i_2, \ldots, i_k)$ is nonincreasing in n and p.

Before concluding this paper, we present further properties of $v_r(i_1, i_2, \ldots, i_k)$.

LEMMA 1.6. Let $(i_1, i_2, \ldots, i_k)$ be the information pattern with $i_{s+1} - i_s > 1$ for some $s (0 \leqq s \leqq k)$, where i_0 and i_{k+1} are interpreted as 0 and r+1. Then, for integer i, $i_s < i < i_{s+1}$,

$$v_r(i_1, \ldots, i_s, i_{s+1}, \ldots, i_k) \leqq v_r(i_1, \ldots, i_s, i, i_{s+1}, \ldots, i_k). \tag{13}$$

(When s=0 or s=k, interprete (13) as $v_r(i_1, \ldots, i_k) \leqq v_r(i, i_1, \ldots, i_k)$ or $v_r(i_1, \ldots, i_k) \leqq v_r(i_1, \ldots, i_k, i)$.)

When the information pattern is ϕ , $v_r(\phi) \leqq v_r(i)$, $1 \leqq i \leqq r$.

Proof. Proof is by induction on r, in the same manner as in Lemma 1.4.

Lemma 1.6 states that $v_r(\cdot)$ does not decrease, given additional information. As an immediate consequence of Lemma 1.6, we have the following result.

LEMMA 1.7. Assume that we are in state $(r;i \mid i_1,i_2,\ldots,i_k)$ with the reference number j. Then it is optimal to make an offer to the r^{th} candidate if

(14) $$g_r(i,j) \geqq v_r(i_1,\ldots,i_j,i,i_{j+1}+1,\ldots,i_k+1).$$

(When j=0 or j=k, interprete (14) as $g_r(i,0)\geqq v_r(i,i_1+1,\ldots,i_k+1)$ or $g_r(i,k)\geqq v_r(i_1,i_2,\ldots,i_k,i)$.)

The above statement also holds in state $(r;i \mid \phi)$ with (14) replaced by $g_r(i,0)\geqq v_r(i)$.

Proof. It is optimal to make an offer to the r^{th} candidate in state $(r;i \mid i_1,i_2,\ldots,i_k)$ if and only if $s_r(i \mid i_1,i_2,\ldots,i_k)\geqq c_r(i \mid i_1,i_2,\ldots,i_k)$, which is, from (2)-(3), equivalent to

(15) $$p\{ g_r(i,j) - v_r(i_1,\ldots,i_j,i_{j+1}+1,\ldots,i_k+1)\}$$
$$\geqq v_r(i_1,\ldots,i_j,i_{j+1}+1,\ldots,i_k+1)$$
$$- v_r(i_1,\ldots,i_j,i,i_{j+1}+1,\ldots,i_k+1).$$

The right side of (15) is nonpositive from Lemma 1.6, so (14) implies (15). The latter part follows similarly.

LEMMA 1.8. Let $(i_1,i_2,\ldots,i_k)$ be the information pattern with $i_{s+1}-i_s>1$ and $i_{t+1}-i_t>1$ for some s and $t(0\leqq s<t\leqq k)$. Then, for integers i_a and i_b, $i_s<i_a<i_{s+1}$ and $i_t<i_b<i_{t+1}$,

$$v_r(i_1,\ldots,i_s,i_a,i_{s+1},\ldots,i_t,i_{t+1},\ldots,i_k)$$
$$\geqq v_r(i_1,\ldots,i_s,i_{s+1},\ldots,i_t,i_b,i_{t+1},\ldots,i_k)$$

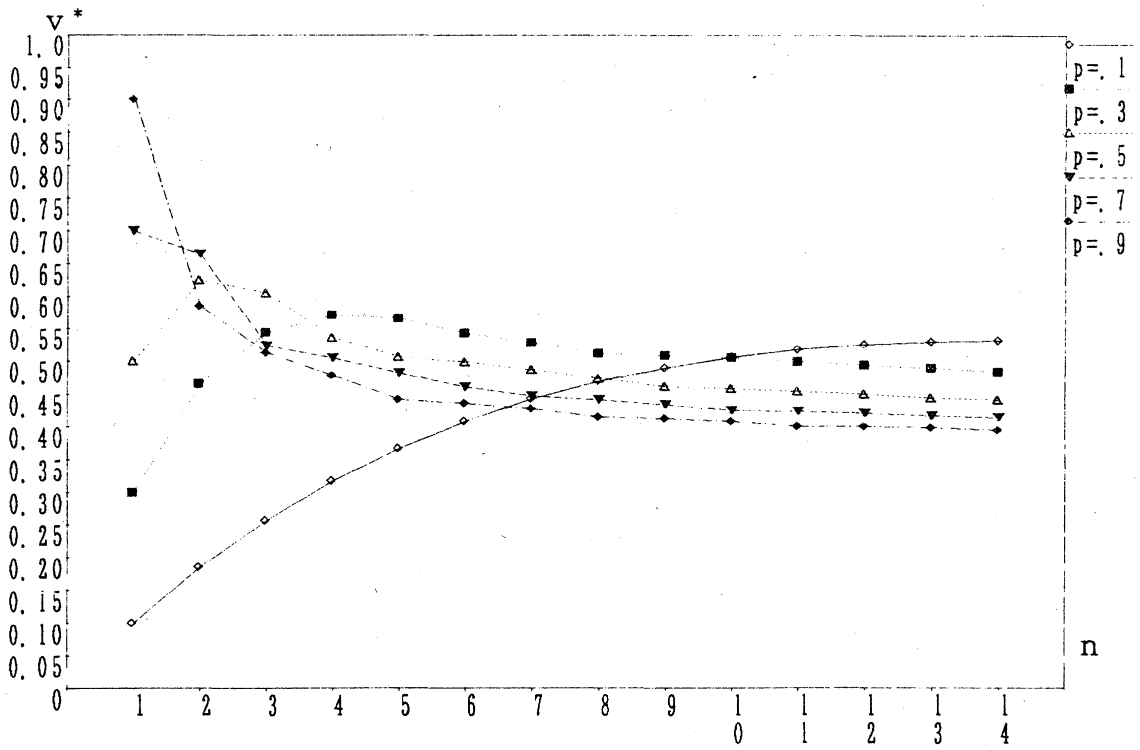
Proof. Since the information pattern $(i_1,\ldots,i_s,i_{s+1},\ldots,i_t,i_b,i_{t+1},\ldots,i_k)$ can be transformed into $(i_1,\ldots,i_s,i_a,i_{s+1},\ldots,i_t,i_{t+1},\ldots,i_k)$ by letting $i_{s+1}\rightarrow i_a$, $i_{s+2}\rightarrow i_{s+1},\ldots,i_t\rightarrow i_{t-1}$, and $i_b\rightarrow i_t$ successively, the result follows from the repeated use of Lemma 1.4.

Table Ⅱ

Probability of success v^* for some values of n and p

n	p				
	0.1	0.3	0.5	0.7	0.9
1	0.10000	0.30000	0.50000	0.70000	0.90000
2	0.18500	0.46500	0.62500	0.66500	0.58500
3	0.25683	0.54450	0.60417	0.52383	0.51300
4	0.31713	0.57113	0.53646	0.50493	0.47835
5	0.36732	0.56617	0.50729	0.48226	0.44163
6	0.40868	0.54338	0.49879	0.46006	0.43592
7	0.44234	0.52898	0.48697	0.44664	0.42666
8	0.46930	0.51216	0.47336	0.44127	0.41384
9	0.49043	0.50924	0.46158	0.43405	0.41246
10	0.50652	0.50563	0.45851	0.42537	0.40817
11	0.51825	0.50048	0.45500	0.42402	0.40168
12	0.52622	0.49496	0.45030	0.42073	0.40133
13	0.53097	0.48970	0.44523	0.41692	0.39896
14	0.53296	0.48471	0.44207	0.41408	0.39542

Figure 1



BIBLIOGRAPHY

1. DeGroot,M.H, *Optimal Statistical Decisions*, McGraw-Hill, New York, 1970.
2. Lindley,D.V, "Dynamic Programming and Decision Theory," Appl. Stat. 10, (1961), 39-51.
3. Petruccelli,J.D, "Full Information Best-choice Problems with Recall of Observations and Uncertainty of Selection Depending on the Observation," Adv.Appl.Prob. 14, (1982), 340-358.
4. Rose,J.S, "The Secretary Problem with a Call Option," Operat. Res.Letters, 3, (1984 a), 237-241.
5. Rose,J.S, "Optimal Sequential Selection Based on Relative Ranks with Renewable Call Options," J.Am.Stat.Assoc. 79, (1984 b), 430-435.
6. Smith,M.H, "A Secretary Problem with Uncertain Employment," J.Appl.Prob. 12, (1975), 620-624.
7. Smith,M.H., and J.J.Deely, "A Secretary Problem with Finite Memory," J.Am.Stat.Assoc. 61, (1975), 357-361.
8. Tamaki,M, "A Full-information Best-choice Problem with Finite Memory," J.Appl.Prob. 23, (1986), 718-735.
9. Yang,M.C.K, " Recognizing the Maximum of a Random Sequence Based on Relative Rank with Backward Solicitation," J.Appl. Prob. 11, (1974), 504-512.