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Determinacy of complex analytic foliation germs without integrating factors

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A codim 1 foliation germ F is defined by a germ of 1-form ω satisfying the integrability condition $d\omega\wedge\omega = 0$ and an unfolding of $F = (\omega)$ is a codim 1 foliation germ on an enlarged space which reduces to the original one when restricted to the original space. In [6] and [8], it is proved that an infinitesimally versal unfolding is versal (versality theorems) and in [7], it is proved that if F is infinitesimally k-detemined, then it is locally k-determined. This article contains some applications of these results. Especially we give some determinacy results for foliation germs without integrating factors.

In section 1, we recall definitions and results concerning the unfolding theory and determinacy of codim 1 complex analytic foliation germs. We prove, in section 2, that for a foliation germ $F = (\omega)$ without formal integrating factors, the finiteness of the dimension of the space $I(\omega)/J(\omega)$ of isomorphism classes of first order unfoldings implies its local finite determinacy ((2.1)Theorem). We also give a sufficient condition for the finite dimensionality of $I(\omega)/J(\omega)$ in terms of the singular sets of ω and of $d\omega$. The singular set of $d\omega$ is also related to the unfolding property of $F = (\omega)$. In section 3, we give as an application of the versality theorem, a simple proof of a theorem of Camacho and Lins Neto asserting that, in dimension 3, if the singular set of $d\omega$ is isolated, then F is stable, i.e., every unfolding of F is trivial. In section 4, we give an effective estimate of the order of determinacy for general foliation germs in dimension 2.

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1. Preliminaries. We recall a theorem of local finite determinacy for codim 1 foliation germs proved in [7]. For this we also need some languages from the unfolding theory for such germs ([6]-[9]).

Let $\mathcal{O}_n$ denote the ring of germs of holomorphic functions at the origin 0 in $\mathbb{C}^n = \{(x_1, \cdots, x_n)\}$ and let Ω_n and Θ_n denote, respectively, the $\mathcal{O}_n$-modules of germs of holomorphic 1-forms and of holomorphic vector fields at 0 in $\mathbb{C}^n$. A codim 1 foliation germ at 0 in $\mathbb{C}^n$ is a rank 1 free sub-$\mathcal{O}_n$-module $F = (\omega)$ of Ω_n with a generator ω satisfying the integrability condition $d\omega \wedge \omega = 0$. The germ at 0 of the analytic set $\{x | \omega(x) = 0\}$ is denoted by $S(\omega)$ or $S(F)$ and is called the singular set of F. We always assume that codim $S(F) \geq 2$.

An unfolding of $F = (\omega)$ is a codim 1 foliation germ $\mathcal{F} = (\tilde{\omega})$ at 0 in $\mathbb{C}^n \times \mathbb{C}^m$, for some m, with a generator $\tilde{\omega}$ such that $\iota^*\tilde{\omega} = \omega$, where ι denotes the embedding of $\mathbb{C}^n = \{x\}$ into $\mathbb{C}^n \times \mathbb{C}^m = \{(x, t)\}$ given by $\iota(x) = (x, 0)$. We call $\mathbb{C}^m$ the parameter space of $\mathcal{F}$. If we also denote by ι_t the embedding of $\mathbb{C}^n$ into $\mathbb{C}^n \times \mathbb{C}^m$ given by $\iota_t(x) = (x, t)$ for t near 0, we may think of $\omega_t = \iota_t^*\tilde{\omega}$ as a deformation of ω. For the notions of a morphism and an RL-morphism, we refer to [7](2.1)Definition or [9](2.1)Definition. For a codim 1 foliation germ $F = (\omega)$, we associate the following algebraic objects ([7]p.432, [8]p.601):

$$I(\omega) = \{h \in \mathcal{O}_n |\ hd\omega = \eta \wedge \omega \quad \text{for some } \eta \text{ in } \Omega_n\},$$

$$J(\omega) = \{h \in \mathcal{O}_n |\ h = i_X\omega \quad \text{for some } X \text{ in } \Theta_n\} \quad \textit{and}$$

$$K(\omega) = \{h \in \mathcal{O}_n |\ hd\omega = dh \wedge \omega\},$$

where i_X denotes the interior product with respect to the vector field X. Also, for a positive integer k, we set

$$I^{(k)}(\omega) = \{h \in \mathcal{O}_n |\ hd\omega + (\theta - dh) \wedge \omega = 0 \text{ for some } \theta \text{ in } \mathfrak{m}^{k-1}\Omega_n\},$$

where $\mathfrak{m}$ denotes the maximal ideal in $\mathcal{O}_n$. Note that $I(\omega)$ is an ideal in $\mathcal{O}_n$ and $J(\omega)$ is a subideal of $I(\omega)$ ([7](2.8)Corollary). They are independent of the chice of the generator ω of F. By setting $I^{(\infty)}(\omega) = K(\omega)$, which is the set of integrating factors of ω, we have a decreasing sequence of $\mathbb{C}$-vector subspaces of $I(\omega)$:

$$I(\omega) = I^{(1)}(\omega) \supset I^{(2)}(\omega) \supset \cdots \supset I^{(k)}(\omega) \supset \cdots \supset I^{(\infty)}(\omega) = K(\omega).$$

If ω' is another generator of F, then $\omega' = u\omega$ for some unit u in $\mathcal{O}_n$. The correspondence $h \mapsto uh$ gives an isomorphism of $I^{(k)}(\omega)$ onto $I^{(k)}(\omega')$ for each $k = 1, 2, \cdots, \infty$. The quotients $I(\omega)/J(\omega)$ and $I(\omega)/J(\omega)+K(\omega)$ are interpreted, respectively, as the sets of isomorphism classes and of RL-isomorphism classes of first order unfoldings of $F=(\omega)$ ([8](1.4)Proposition).

For a germ g in $\mathcal{O}_n$, we denote by $j^k g$ the k-jet of g and for a germ $\theta=\sum_{i=1}^{n} g_i dx_i$ in Ω_n, we define the k-jet $j^k\theta$ of θ by

$$j^k\theta=\sum_{i=1}^{n} j^{k-1}g_i\cdot dx_i.$$

(1.1)DEFINITION: An unfolding $\mathcal{F}$ of a codim 1 foliation germ $F=(\omega)$ with parameter space $\mathbb{C}^m=\{t\}$ is k-trivial if it has a generator $\tilde{\omega}$ such that $\iota^*\tilde{\omega}=\omega$ and that $j^k\omega_t=j^k\omega$ for all t near 0 in $\mathbb{C}^m$.

We say that two germs ω and ω' in Ω_n are holomophically equivalent and write $\omega\sim\omega'$ if there exist a germ at 0 of biholomorphic map φ of $\mathbb{C}^n$ into itself with $\varphi(0)=0$ and a unit u in $\mathcal{O}_n$ such that $\omega'=u\varphi^*\omega$. If ω is integrable and if $\omega'\sim\omega$, ω' is also integrable and ω and ω' define isomorphic foliations.

(1.2)DEFINITION: A codim 1 foliation germ $F(\omega)$ is locally k-determined if for every k-trivial unfolding $\mathcal{F}=(\tilde{\omega})$ of F, we have $\omega_t\sim\omega$ for all t near 0.

(1.3)DEFINITION: A codim 1 foliation germ $F=(\omega)$ is infinitesimally k-determined if

$$I^{(k+1)}(\omega)\subset \mathfrak{m}J(\omega)+K(\omega).$$

For the geometric meaning of the above condition, see [7]p.437. We quote the following

(1.4)THEOREM([7]). *Let $F=(\omega)$ be a* codim 1 *foliation germ with*

(*) $$\dim\ K(\omega)/\mathfrak{m}J(\omega)\cap K(\omega)<+\infty.$$

If F is infinitesimally k-determined, then it is locally k-determined.

(1.5) REMARK: Actually, in [7] it is shown that, under the assumption (*), if F is infinitesimally k-determined, then every k-trivial (one parameter) undolfing of F is (strongly) RL-trivial. Moreover, when $K(\omega) = 0$, every k-trivial (one parameter) unfolding of F is (strongly) trivial.

2. Determinacy of foliation germs without formal integrating factors.

A formal integrating factor of an integrable 1-form ω is a formal power series $\hat{h}$ in $x = (x_1, \cdots, x_n)$ such that $\hat{h}d\omega = d\hat{h} \wedge \omega$. We say that a codim 1 foliation germ $F = (\omega)$ is locally finitely determined if it is locally k-determined for some k.

(2.1) THEOREM. *Let $F = (\omega)$ be a* codim 1 *foliation germ at* 0 *in $\mathbb{C}^n$ without (non-zero) formal integrating factors. If the $\mathbb{C}$-vector space $I(\omega)/J(\omega)$ is finitely dimensional, then F is locally finitely determined.*

To prove this, we first prove the following

(2.2) LEMMA. *Let $F = (\omega)$ be a* codim 1 *foliation germ without formal integrating factors. For every positive integer k, there exists another k' such that*

$$I^{(k')}(\omega) \subset \mathfrak{m}^k \cap I(\omega).$$

PROOF: If $I^{(k)}(\omega) = \{0\}$ for some k, then there is nothing to be proved. Thus we assume that $I^{(k)}(\omega) \neq \{0\}$ for all k. For each k, we set

$$n(k) = \min\{\nu(h) | h \in I^{(k)}(\omega)\},$$

where $\nu(h)$ denotes the order of h. We have an increasing sequence $\{n(k)\}$ of non-negative integers. For our purpose, it suffices to show that

$$\lim_{k \to \infty} n(k) = +\infty.$$

Suppose the sequence is bounded and let n_0 be an integer such that $n(k) \leq n_0$ for all k. Then $j^{n_0} I^{(k)}(\omega) \neq \{0\}$ for all k. We have a decreasig sequence of finite dimensional vector spaces

$$\cdots \supset j^{n_0} I^{(k)}(\omega) \supset j^{n_0} I^{(k+1)}(\omega) \supset \cdots \supsetneq \{0\}.$$

Thus there exists k_0 such that for all $k \geq k_0$,

$$j^{n_0} I^{(k)}(\omega) = j^{n_0} I^{(k_0)}(\omega).$$

We fix a germ H in $I^{(k_0)}(\omega)$ with $j^{n_0} H \neq 0$. Then for all k, there exists h in $I^{(k)}(\omega)$ such that $j^{n_0} h = j^{n_0} H$. We claim that there exists a non-zero formal power series $\hat{h}$ in x such that for all n, we have

$(2.3)_n$ for all k, there exists h in $I^{(k)}(\omega)$ with $j^n h = j^n \hat{h}$.

We determine the n-jet of $\hat{h}$ for every n by induction. If we let the n_0-jet of $\hat{h}$ be equal to $j^{n_0} H$, then we have $(2.3)_n$ for $n \leq n_0$. Suppose we have determined the n-jet of $\hat{h}$ so that $(2.3)_n$ is satisfied. We set

$$I^{(k)}(\omega)_n = \{h \in I^{(k)}(\omega) |\ j^n h = j^n \hat{h}\},$$

which is a (non-empty) affine subspace of $I(\omega)$. We have a decreasing sequence of finite dimensional affine spaces

$$\cdots \supset j^{n+1} I^{(k)}(\omega)_n \supset j^{n+1} I^{(k+1)}(\omega)_n \supset \cdots.$$

Thus there exists k_1 such that for all $k \geq k_1$,

$$j^{n+1} I^{(k)}(\omega)_n = j^{n+1} I^{(k_1)}(\omega)_n.$$

Hence there exists a homogeneous polynomial h_{n+1} of degree $n+1$ such that $j^n \hat{h} + h_{n+1}$ is in $j^{n+1} I^{(k)}(\omega)$ for all k. If we let the $n+1$ jet of $\hat{h}$ be equal to $j^n \hat{h} + h_{n+1}$, then we have $(2.3)_{n+1}$.

Clearly the formal power series $\hat{h}$ thus constructed is a non-zero formal integrating factor of ω.

Proof of (2.1)**Theorem**: By (1.4)Theorem, it suffices to show that $I^{(k)}(\omega) \subset \mathfrak{m}J(\omega)$ for some k. First, by the finite dimensionality of $I(\omega)/J(\omega)$, we have

$$\mathfrak{m}^k I(\omega) \subset \mathfrak{m}J(\omega)$$

for some k. Second, by the Artin-Rees theorem (e.g.[5]), we have

$$\mathfrak{m}^{k'} \cap I(\omega) \subset \mathfrak{m}^k \cdot I(\omega)$$

for some k'. Finally by (2.2)Lemma, we have

$$I^{(k'')}(\omega) \subset \mathfrak{m}^{k'} \cap I(\omega)$$

for some k''. Q.E.D.

When $n = 2$, by the assumption codim $S(\omega) \geq 2$, $\mathcal{O}_2/J(\omega)$ is always finite dimensional. Thus we have

(2.4)Corollary. *Every* codim 1 *foliation germ at* 0 *in* $\mathbb{C}^2$ *without formal integrating factors is locally finitely determined.*

(2.5)Remarks: 1°. Let $F = (\omega)$ be a codim 1 foliation germ without formal integrating factors. The above arguments show that if $I(\omega)/J(\omega)$ is finite dimensional, then F is infinitesimally finitely determined. The converse is also true, since $K(\omega) = \{0\}$ in this case (see [7](4.3)Remark).

2°. Let $F = (\omega)$ be a codim 1 foliation germ. If ω admits a non-zero holomorphic integrating factor, then ω admits a multiform first integral ([3]Théorème d'intégration).

3°. As for the determinacy problem for multiform functions see [3] Sixième partie and [7]§5.

Here is a case when we have the finite dimensionality of $I(\omega)/J(\omega)$. Let $F = (\omega)$ be a codim 1 foliation germ. If we let $S(d\omega)$ be (the germ at 0 of) the zero set of $d\omega$, we easily see that $S(\omega) \cap S(d\omega)$ does not depend on the choice of the generator ω of F.

(2.6)PROPOSITION. *Let* $F = (\omega)$ *be a* codim 1 *foliation germ. If* $S(\omega) \cap S(d\omega) \subset \{0\}$, *then the vector space* $I(\omega)/J(\omega)$ *is finite dimensional.*

PROOF: The finite dimensionality of $I(\omega)/J(\omega)$ is equivalent to saying that the support of the coherent sheaf $I(\omega)/J(\omega)$ is contained in $\{0\}$. Take a point $x \neq 0$ near 0. By the assumption we have either $\omega(x) \neq 0$ or $d\omega(x) \neq 0$. Suppose $\omega(x) \neq 0$. Then $J(\omega)_x = I(\omega)_x = \mathcal{O}_x$. Suppose $d\omega(x) \neq 0$. For a germ h in $I(\omega)_x$, we have $hd\omega = \eta \wedge \omega$ for some η in Ω_x. Since $d\omega(x) \neq 0$, h is in $J(\omega)_x$.

3. Stability of simple forms in $\mathbf{C}^3$. Let ω be a germ of integrable 1-form. As we noted in the previous section, the size of the zero set of $d\omega$ seems to have some effect on the determinacy of ω. It is also related to the unfolding property of ω. In Camacho-Lins Neto [1], a germ ω at 0 in $\mathbf{C}^3$ is called simple if $S(d\omega) \subset \{0\}$ and it is proved that a simple form ω is stable, i.e., every unfolding of ω is trivial ([1]Theorem 4). In this section, we give a simple proof of this result as a direct application of the versality theorem in [6].

(3.1)THEOREM. *Let* ω *be a germ of integrable* 1-form *at* 0 *in* $\mathbf{C}^3$. *If* $S(d\omega) \subset \{0\}$, *then* $I(\omega)/J(\omega) = 0$. *Thus every unfolding of the foliation* $F = (\omega)$ *is trivial.*

PROOF: First, we proceed as in the proof of Theorem 1 in [1]. Thus, denoting by (x, y, z) coordinates of $\mathbf{C}^3$, we let X be the vector field germ such that

$$d\omega = i_X(dx \wedge dy \wedge dz).$$

From the integrability of ω, we easily see that $i_X\omega = 0$. Hence, by the assumption on $S(d\omega)$, we may write $\omega = i_X\tau$ for some 2-form τ. If we let Y be the vector field such that $\tau = i_Y(dx \wedge dy \wedge dz)$, we get

$$\omega = i_X i_Y(dx \wedge dy \wedge dz).$$

Second, we let $F = (\omega)$ be the $\mathcal{O}$-module generated by ω and let F^a be the annihilator of F in Θ:

$$F^a = \{Z \in \Theta | i_Z\omega = 0\}.$$

Obviously the vector fields X and Y are in F^a and generate F^a outside of $S(\omega)$. Since codim $S(\omega) \geq 2$, they also generate F^a across $S(\omega)$.

Third, we take a germ h in $I(\omega)$. Then $hd\omega = \eta \wedge \omega$ for some η in Ω. By assumption, for any point x ($\neq 0$) near 0, $d\omega(x) \neq 0$. Hence the germ h_x of h at x is in $J(\omega)_x$. We take a small Stein open neighborhood U of 0 in $\mathbb{C}^3$. There is a covering $\mathcal{U} = \{U_\alpha\}$ of $U\backslash\{0\}$ by small open sets U_α such that on each U_α, we may write

$$h = i_{X_\alpha}\omega$$

for some vector field X_α on U_α. In the intersection $U_\alpha \cap U_\beta$, we have $i_{X_\alpha - X_\beta}(\omega) = 0$. Hence $X_\alpha - X_\beta$ is in F^a and we may write

$$X_\alpha - X_\beta = f_{\alpha\beta}X + g_{\alpha\beta}Y$$

for holomorphic functions $f_{\alpha\beta}$ and $g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. We have the cocycles $\{f_{\alpha\beta}\}$ and $\{g_{\alpha\beta}\}$ which determine cohomology classes in $H^1(\mathcal{U}, \mathcal{O}) = H^1(U\backslash\{0\}, \mathcal{O})$. Since this cohomology group vanishes, we may write

$$f_{\alpha\beta} = f_\beta - f_\alpha \quad \text{and} \quad g_{\alpha\beta} = g_\beta - g_\alpha$$

for holomorphic functions f_α and g_α on U_α. Thus we have a vector field Z on $U\backslash\{0\}$, which is given by

$$Z = X_\alpha + f_\alpha X + g_\alpha Y \quad \text{on} \quad U_\alpha.$$

We have

$$h = i_Z\omega \tag{3.2}$$

on $U\backslash\{0\}$. The vector field Z extends across 0 and we have (3.2) in $\mathcal{O}$. Hence h is in $J(\omega)$ and we proved that $I(\omega)/J(\omega) = 0$. By [6](4.1)Corollary, every unfolding of $F = (\omega)$ is trivial. Q.E.D.

4. Effective estimate of the order of determinacy. In this section, we consider foliation germs at 0 in $\mathbb{C}^2 = \{(x, y)\}$. We denote by Ω^p_k the vector space of homogeneous

p-forms of degree k on $\mathbb{C}^2$. The space Ω^0_k of homogeneous polynomials of degree k in (x,y) is especially denoted by $\mathcal{O}_k$. Thus we may write $\Omega^1_k = \mathcal{O}_{k-1}dx \oplus \mathcal{O}_{k-1}dy$ and $\Omega^2_k = \mathcal{O}_{k-2}dx \wedge dy$. Note that our definition of the degree of a form is different from the conventional one.

Let ω_{k_0} be an element in $\Omega^1_{k_0}$. Denoting by R the radial vector field;

$$R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

we set

$$P_{k_0} = i_R\omega_{k_0}.$$

Hereafter we only consider the case $P_{k_0} \neq 0$, i.e., ω_{k_0} is non-dicritic. We easily see that P_{k_0} is an integrating factor of ω_{k_0}.

(4.1)DEFINITION: ([3]p.134, Cf. Also [4] and [2]). We say that ω_{k_0} is generic if

(a) $S(\omega_{k_0}) \subset \{0\}$.

(b) P_{k_0} is reduced.

(c) ω_{k_0} does not admit meromorphic first integrals, i.e., there is no non-constant meromorphic function germ φ with $d\varphi \wedge \omega_{k_0} = 0$.

Suppose ω_{k_0} is generic. Since the quotient of two integrating factors of ω_{k_0} is a meromorphic first integral of ω_{k_0}, the condition (c) implies that

$$K(\omega_{k_0}) = \mathbb{C}P_{k_0}. \tag{4.2}$$

We set

$$V(\omega_{k_0}) = \{\theta \in \Omega^1_{k_0+1} | P_{k_0}d\theta - dP_{k_0} \wedge \theta = df \wedge \omega_{k_0} - fd\omega_{k_0} \text{ for some } f \text{ in } \mathcal{O}_{k_0+1}\}.$$

(4.3)LEMMA. *The set $V(\omega_{k_0})$ is a sub-vector space of $\Omega^1_{k_0+1}$ of codimension at least $2k_0$-$\min\{2k_0, k_0+2\}$.*

Proof: We consider two linear maps

$$\lambda : \Omega^1_{k_0+1} \to \Omega^2_{2k_0+1} \qquad \text{and}$$
$$\mu : \mathcal{O}_{k_0+1} \to \Omega^2_{2k_0+1}$$

defined by $\lambda(\theta) = P_{k_0}d\theta - dP_{k_0} \wedge \theta$ for θ in $\Omega^1_{k_0+1}$ and by $\mu(f) = df \wedge \omega_{k_0} - fd\omega_{k_0}$ for f in $\mathcal{O}_{k_0+1}$. By [3] Théorème d'intégration, we see that

$$\text{Ker } \lambda = \{\theta \in \Omega^1_{k_0+1} | \theta = P_{k_0}dL \text{ for some } L \text{ in } \mathcal{O}_1\}.$$

Hence rank $\lambda = 2(k_0+1) - 2 = 2k_0$. On the other hand, by (4.2), Ker $\mu = 0$. Hence rank $\mu = k_0 + 2$. Thus we have

$$\begin{aligned}\dim V(\omega_{k_0}) &= \dim \text{ Ker } \lambda + \dim(\text{Im } \lambda \cap \text{Im } \mu)\\ &\leq 2 + \min\{2k_0, k_0+2\}.\end{aligned}$$

(4.4) **Theorem.** *Let ω be a germ of integrable 1-form at 0 in $\mathbb{C}^2$ and let ω_{k_0} be the first non-zero jet of ω. Further, we let $\omega_{k_0+1} = j^{k_0+1}\omega - \omega_{k_0}$. Suppose $k_0 \geq 3$, ω_{k_0} is generic and ω_{k_0+1} is not in $V(\omega_{k_0})$. Then*

$$I^{(k)}(\omega) = \mathfrak{m}^k \cap I(\omega)$$

for $k > k_0 + 1$.

Proof: For a germ of holomorphic function h and a germ of holomorphic 1-form θ, we set $h_k = j^k h - j^{k-1}h$ and $\theta_k = j^k\theta - j^{k-1}\theta$. In what follows, we fix k with $k > k_0 + 1$. If h is a germ in $I^{(k)}(\omega)$, we have

$$hd\omega + (\theta - dh) \wedge \omega = 0 \tag{4.5}$$

for some θ with $j^{k-1}\theta = 0$. From this, we easily see that

$$j^{k_0-1}h = 0 \quad \text{and} \quad h_{k_0} \in K(\omega_{k_0}).$$

Thus by (4.2), we may write $h_{k_0} = cP_{k_0}$ for some c in $\mathbb{C}$. Comparing the terms of degree $2k_0 + 1$ in (4.5), we get

$$c(P_{k_0} d\omega_{k_0+1} - dP_{k_0} \wedge \omega_{k_0+1}) = dh_{k_0+1} \wedge \omega_{k_0} - h_{k_0+1} d\omega_{k_0}.$$

Since ω_{k_0+1} is not in $V(\omega_{k_0})$, we have $c = 0$. Thus $j^{k_0}h = 0$. Then, by (4.5) and (4.2), we see that

$$j^{k_0+1}h = \cdots = j^{k-1}h = 0.$$

Hence we have $I^{(k)}(\omega) \subset \mathfrak{m}^k \cap I(\omega)$. Conversely, let h be a germ in $\mathfrak{m}^k \cap I(\omega)$. We have the identity (4.5) for some θ in Ω. By the condition (a) in (4.1), if $\eta \wedge \omega_{k_0} = 0$ for η in Ω, then $\eta = g\omega_{k_0}$ for some g in $\mathcal{O}$. Thus from (4.5), we see that

$$j^{k_0-1}\theta = 0.$$

Comparing the terms of degrees from $2k_0$ to $k_0 + k - 1$, we see that for each ℓ with $0 \leq \ell \leq k - k_0 - 1$, there is an element g_ℓ in $\mathcal{O}_\ell$ such that

$$\theta_\ell = \sum_{\ell'=k_0}^{\ell} g_{\ell-\ell'}\omega_{\ell'}$$

for ℓ with $k_0 \leq \ell \leq k-1$. If we set $g = \sum_{\ell=0}^{k-k_0-1} g_\ell$ and $\theta' = \theta - g\omega$, then we have

$$hd\omega + (\theta' - dh) \wedge \omega = 0 \quad \text{and} \quad j^{k-1}\theta' = 0.$$

Hence h is in $I^{(k)}(\omega)$.

(4.6) **Remark**: The above theorem holds also for $k = \infty$, i.e., for a germ ω as in (4.4) Theorem, $K(\omega) = 0$. We also have $K(j^k\omega) = 0$ for $k > k_0$.

(4.7) **Corollary.** *Let ω be as in (4.4) Theorem. If*

$$\mathfrak{m}^{k+1} \cap I(\omega) \subset \mathfrak{m}J(\omega)$$

for some $k > k_0$, then $F = (\omega)$ is locally k-determined.

(4.8)REMARK: By the Artin-Rees theorem, there is always k for which the condition in (4.7) is satisfied.

(4.9)EXAMPLE: Let ω be as in (4.4)Theorem and assume $k_0 = 3$. Then we easily see that $\mathfrak{m}^3 \subset J(\omega_3)$. On the other hand, by the Nakayama lemma, we have $J(\omega) = J(\omega_3)$. Hence

$$\mathfrak{m}^{4+1} \subset \mathfrak{m}^4 \subset \mathfrak{m}J(\omega).$$

Thus $F = (\omega)$ is locally 4-determined.

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