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NORM INEQUALITIES EQUIVALENT TO LÖWNER-HEINZ THEOREM

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ABSTRACT. We give several norm inequalities equivalent to the famous Löwner-Heinz inequality.

THEOREM. If A and B are positive bounded linear operators on a Hilbert space, then the following properties hold and follow from each other.

(1) $A\geq B\geq 0$ ensures $A^s\geq B^s$ for any $1\geq s\geq 0$.

(2) $\|AB\|^q\leq\|A^qB^q\|$ for any $q\geq 1$, namely $\|A^qB^q\|^{1/q}\leq\|A^pB^p\|^{1/p}$ for any $p\geq q>0$,
that is, $f(p)=\|A^pB^p\|^{1/p}$ is an increasing function on p.

(3) $\|A^sB^s\|\leq\|AB\|^s$ for any $1\geq s\geq 0$, namely $\|A^{1/s}B^{1/s}\|^s\leq\|A^{1/t}B^{1/t}\|^t$ for any $s\geq t>0$,
that is, $g(s)=\|A^{1/s}B^{1/s}\|^s$ is a decreasing function on s.

(4) $\|AB\|^{(p+q)/2}\leq\|A^pB^p\|^{1/2}\|A^qB^q\|^{1/2}$ for any $p\geq 0$, $q\geq 0$ with $(p+q)/2\geq 1$.

(5) $\|A^{st}B^{st}\|^2\leq\|A^sB^s\|^{2st/(s+t)}\|A^tB^t\|^{2st/(s+t)}$ for any $s>0$, $t>0$ with $2st/(s+t)\leq 1$.

(6) $\|AB\|^{(p+q)/2}\leq\|A^pB^q\|^{1/2}\|A^qB^p\|^{1/2}$ for any $p\geq 0$, $q\geq 0$ with $(p+q)/2\geq 1$.

(7) $\|A^{st}B^{st}\|^2\leq\|A^sB^t\|^{2st/(s+t)}\|A^tB^s\|^{2st/(s+t)}$ for any $s>0$, $t>0$ with $2st/(s+t)\leq 1$.

We remark that (1) has been shown in [cf., [3][4][5][6] etc.] and (3) is shown in [2]. Here we state the following lemma.

Lemma. If A and B are positive bounded linear operator on a Hilbert space, then

$$\|A^{(s+t)/2}B^{(s+t)/2}\|^2\leq\|B^tA^{s+t}B^s\| \text{ for any } s\geq 0 \text{ and } t\geq 0.$$

Proof of Lemma.

$$\|A^{(s+t)/2}B^{(s+t)/2}\|^2 = \|B^{(s+t)/2}A^{s+t}B^{(s+t)/2}\|$$

$$= r(B^{(s+t)/2}A^{s+t}B^{(s+t)/2}) \text{ (r(A) means the spectral radius of A)}$$

$$= r(B^tA^{s+t}B^s) \quad \text{since } r(AB) = r(BA) \text{ for any A and B}$$

$$\leq \|B^tA^{s+t}B^s\|.$$

Proof of Theorem.

Proof of (3). Here we give an alternative proof to (3). Put $D = \{s \in [0,1] \,;\, \|A^sB^s\| \leq \|AB\|^s\}$. Then D is a closed set such that $0, 1 \in D$, so we have only to show that if $s, t \in D$, then $(s+t)/2 \in D$.

$$\|A^{(s+t)/2}B^{(s+t)/2}\|^2 \leq \|B^tA^{s+t}B^s\| \quad \text{by Lemma}$$

$$\leq \|B^tA^t\|\,\|A^sB^s\|$$

$$\leq \|AB\|^t\|AB\|^s = \|AB\|^{s+t} \quad \text{since } s, t \in D,$$

so that $\|A^{(s+t)/2}B^{(s+t)/2}\|^2 \leq \|AB\|^{(s+t)/2}$, that is, $(s+t)/2 \in D$, whence we have (3).

(2)$\longleftrightarrow$(3). Its proof is obvious.

(3)$\longleftrightarrow$(1). We may assume that A and B are invertible. Assume (3). The condition (3) is equivalent to the following (8) by the homogeneity of norm

(8) $\qquad \|AB\| \leq 1$ ensures $\|A^sB^s\| \leq 1$ for any $1 \geq s \geq 0$.

By replacing A by $A^{-1/2}$ and also B by $B^{1/2}$ in (8), this condition (8) means that $A^{-1/2}BA^{-1/2} \leq 1$ ensures $A^{-s/2}B^sA^{-s/2} \leq 1$, that is, $A \geq B \geq 0$ ensures $A^s \geq B^s$ for any $1 \geq s \geq 0$, so we have (3)$\longrightarrow$(1). Conversely assume (1). (1) is equivalent to the following (9)

(9) $\qquad \|A^{-1/2}B^{1/2}\| \leq 1$ ensures $\|A^{-s/2}B^{s/2}\| \leq 1$ for any $1 \geq s \geq 0$.

By replacing A by A^{-2} and also B by B^2 in (9), so we have (8) and this condition (8) is equivalent to (3), so we have (1)$\longrightarrow$(3).

Whence we have (1)$\longleftrightarrow$(3).

(2)$\longrightarrow$(4) and (6). Assume (2). Then

$$\|AB\|^{p+q} \le \|A^{(p+q)/2}B^{(p+q)/2}\|^2 \quad \text{by (2) since } (p+q)/2 \ge 1$$

$$\le \|B^pA^{p+q}B^q\| \quad \text{by Lemma}$$

$$\le \|B^pA^p\|\,\|A^qB^q\| \quad \text{or } \|B^pA^q\|\,\|A^pB^q\|$$

whence we have (4) and (6).

(4) or (6)$\longrightarrow$(2). Put $p = q$ in (4) or (6), then we have (2).

(4)$\longleftrightarrow$(5) and (6)$\longleftrightarrow$(7). Put $s = 1/p$ and $t = 1/q$ in (4) and (6) and also replace A by A^{st} and B by B^{st}, then we have (5) and (7) and the reverse implications are obvious.

Whence the proof of Theorem is complete.

REMARK. Related to (2) it is easily verified that $\|AB\|^q \le \|A^qB^q\|$ does not always hold for $1 > q > 0$. Related to this result we would like to remark the following result. Put $h(p) = \|A^pB^p\|/\|AB\|^p$ for any $p \ge 0$. (4) asserts that $h(p)h(q) \ge 1$ for any $p \ge 0$, $q \ge 0$ with $p+q \ge 2$. Put $A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $h(1/2) = \sqrt{5}/34^{1/4} < 1$ and $h(3/2) = \sqrt{233}/34^{3/4} > 1$ but $h(1/2)h(3/2) = \sqrt{1165}/34 > 1$.

Acknowledgment. On December 6, 1988 Professor H. Araki has given an excellent lecture at the meeting of Operator Theory at the Research Institute of Mathematical Sciences of Kyoto University ([1]). We became aware of the importance of (2) in his lecture. We are really impressed with his instructive topics, so here we have been able to give a short proof to (3) by using the idea of Pedersen ([6]). We would like to express our thanks to Professor H. Araki for his attractive topics.

Addendum. Recently we have the following results in [8] as an application of [7].

THEOREM 1. If A and B are arbitrary bounded linear operators on a Hilbert space, then the following properties hold and follow from each other.

(1) $A \geq B \geq 0$ ensures $A^s \geq B^s$ for any $1 \geq s \geq 0$.

(2) $\|AB\|^q \leq \| |A|^q |B^*|^q \|$ for any $q \geq 1$, namely $\| |A|^q |B^*|^q \|^{1/q} \leq \| |A|^p |B^*|^p \|^{1/p}$ for any $p \geq q > 0$, that is, $f(p) = \| |A|^p |B^*|^p \|^{1/p}$ is an increasing function on p.

(3) $\| |A|^s |B^*|^s \| \leq \|AB\|^s$ for any $1 \geq s \geq 0$, namely $\| |A|^{1/s} |B^*|^{1/s} \|^s \leq \| |A|^{1/t} |B^*|^{1/t} \|^t$ for any $s \geq t > 0$, that is, $g(s) = \| |A|^{1/s} |B^*|^{1/s} \|^s$ is a decreasing function on s.

(4) $\|AB\|^{(p+q)/2} \leq \| |A|^p |B^*|^p \|^{1/2} \| |A|^q |B^*|^q \|^{1/2}$ for any $p \geq 0$, $q \geq 0$ with $(p+q)/2 \geq 1$.

(5) $\|AB\|^{(p+q)/2} \leq \| |A|^p |B^*|^q \|^{1/2} \| |A|^q |B^*|^p \|^{1/2}$ for any $p \geq 0$, $q \geq 0$ with $(p+q)/2 \geq 1$.

Definition 1. An operator T is said to be perinormal if

$$(T^*T)^n \leq T^{*n} T^n$$

holds for every natural number n. Our new class of perinormal operators occupies the place shown in the following schema and the inclusions are all proper.

$$\text{Normal} \subsetneqq \text{Quasinormal} \subsetneqq \text{Heminormal} \subsetneqq \text{Perinormal} \subsetneqq \text{Normaloid}$$

Theorem 2. If A and B^* are perinormal, then (*) holds;

(*) the following (1) and (2) hold and follow from each other.

(1) $\|AB\|^n \leq \|A^n B^n\|$ for every natural number n.

(2) $\|AB\|^{n+m} \leq \|A^n B^m\| \|A^m B^n\|$ for every natural number n and m.

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