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Navier-Stokes 方程式の局所正則性判定条件

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Part 1

ON INTERIOR REGULARITY CRITERIA FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

We are concerned with the behavior of weak solutions of the Navier-Stokes equations near possible singularities. We shall show that if a weak solution is in some Lebesgue space or small in some Lorentz space locally, it does not blowup there. Our basic idea is to estimate integral formulas for vorticity which satisfies parabolic equations.

1. Introduction

This paper studies local interior regularity criteria for weak solutions of the Navier-Stokes equations:

(1.1) $$u_t - \Delta u + (u\cdot\nabla)u + \nabla\phi = 0 \qquad \text{in } Q$$

(1.2) $$\nabla\cdot u = 0 \qquad \text{in } Q$$

(1.3) $$u|_{\partial\Omega} = 0,\ u(x,0) = u_0,$$

where $Q = \Omega\times(0,T)$, Ω is a domain in $\mathbb{R}^n (n\geq 3)$ with smooth boundary, $0 < T < \infty$; $u = (u^i)_{i=1}^n$ and ϕ denote, respectively, unknown velocity and pressure, while $u_0 = (u_0^i)_{i=1}^n$ is a given initial velocity. Here external force is assumed to be zero for simplicity. For every

$u_0 \in L^2(\Omega)$ satisfying compatibility conditions, a global weak solution was constructed by Leray [Le] (when $\Omega = \mathbb{R}^3$) and Hopf [Ho]. Their solutions are known to satisfy

$$u \in L^{2,\infty}(Q) \quad \text{and} \quad \nabla u \in L^{2,2}(Q) \tag{1.4}$$

where

$$L^{p,q}(Q) = L^q(0,T; L^p(\Omega)).$$

However, the regularity of their weak solutions is not known unless $n = 2$ although some partial regularity is proved for $n = 3$ (see [CKN] and references therein).

Serrin [Se] gave a nice local interior regularity criterion (cf. [Oh]). Let us recall his result. He proved among other results, that a weak solution u satisfying (1.4) is in $L^{\infty,\infty}(Q_{R/2})$ and regular in space variables provided that u satisfies $u \in L^{p,q}(Q_R)$ with

$$n/p + 2/q < 1, \; n < p < \infty. \tag{1.5}$$

Here $Q_R = Q_R(x_0, t_0)$ is a parabolic ball centered at $(x_0, t_0) \in Q$:

$$Q_R(x_0,t_0) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R};\; x \in B_R(x_0), -R^2 < t - t_0 < 0\}$$

such that $Q_R \subset Q$ where $B_R(x_0) = \{x \in \mathbb{R}^n;\; |x - x_0| < R\}$.

Recently Struwe [St] refined Serrin's result allowing the case

$$n/p + 2/q = 1, \; n < p \leq \infty. \tag{1.6}$$

The global version is known by Sohr [So] and Giga [Gi] when $p < \infty$. Indeed, if $u \in L^{p,q}(Q)$ solves the initial-boundary problem of the Navier-Stokes equations (1.1)-(1.3) with (1.5) or (1.6), u is regular in space-time up to the boundary.

Our goal is to give a new interior regularity criterion for (1.1)-(1.2). We prove among other results, that there is $\varepsilon > 0$ such that

$$\sup_{x \in B_R(x_0)} |u(x,t)| \leq \varepsilon (t_0 - t)^{-1/2} \quad \text{for } -R^2 + t_0 < t < t_0 \tag{1.7}$$

implies $u \in L^{\infty,\infty}(Q_{R/2})$. Here ε is independent of u, R and (x_0, t_0). In other words (x_0, t_0) can not be a blowup point if (1.7) holds. Similar results are known for a semilinear heat equation

$$u_t - \Delta u - |u|^{p-1}u = 0 \quad \text{for} \quad p > 1$$

by Giga-Kohn [GK]. Our basic idea is estimating integral formulas for vorticity $\omega = \text{curl}\, u$. This idea goes back to Serrin [Se] while Struwe's proof is based on an energy method. We will show that our method also recovers Struwe's interior regularity criterion. In [St, p.440] Struwe observed that his results may be obtained by a simple extention of Serrin's original method but the details are not explained there. We take this opportunity to present Serrin's approach to get Struwe's result since it is obtained in parallel with our main new regularity criterion (1.7). Since we avoid to use traces in Sobolev spaces of minus exponents which appear in [St], our proof simplifies that of [St] in this respect.

The crucial part of our argument is regularity of solutions of a parabolic system

$$\omega_t - \Delta\omega + \nabla b\omega = 0 \qquad \text{in } Q \tag{1.8}$$

with nonregular coefficient b. We state our main results on (1.8) in Section 2 and results on Navier-Stokes equations in Section 3 including (1.7) where we use Lorentz spaces.

2. Interior Regularity for Parabolic Equations

We consider a parabolic system

$$\omega_t - \Delta\omega + \nabla b\omega = 0 \tag{2.1}$$

in $Q = \Omega \times (0, T)$, where Ω is a domain in $\mathbb{R}^n$ with smooth boundary and $0 < T < \infty$. Here

$$\begin{aligned} &\omega = (\omega^1, \dots, \omega^d) \text{ with } \omega^i = \omega^i(x,t) \quad (i = 1, \dots, d), \\ &b(x,t) = (b^i_{jk}(x,t)) \text{ for } 1 \le i, k \le d \text{ and } 1 \le j \le n, \text{and} \\ &\nabla b\omega = \left(\sum_{j=1}^{n} \sum_{k=1}^{d} \frac{\partial}{\partial x_j} b^i_{jk}(x,t)\omega^k(x,t) \right)_{i=1}^{d}. \end{aligned} \tag{2.2}$$

We shall study a regularity of ω under minimal regularity assumptions on b. Let $L^{p,q}(Q)$ denote the space of $L^p(\Omega)$-valued L^q functions on $(0,T)$. The space $L^{p,q}(Q)$ is equipped with the norm

$$\|u\|_{L^{p,q}(Q)} = [\ \|u\|_{L^p(\Omega)}(t)\ ]_{L^q(0,T)} = \left\{ \int_0^T \left(\int_\Omega |u(x,t)|^p dx \right)^{q/p} dt \right\}^{1/q}.$$

Here $\|\cdot\|_{L^p(\Omega)}$ denotes the space L^p- norm, and $[\ \cdot\ ]_{L^q(0,T)}$ denotes the time L^q- norm. We do not distinguish the spaces of vector and scalar valued functions.

We say $\omega \in L^{2,2}(Q)$ is a *weak solution* of (2.1) in Q, if it holds

$$\iint_Q (\varphi_t + \Delta\varphi + b\nabla\varphi)\omega \ dxdt = 0$$

for any $\varphi \in C_0^\infty(Q)$ where $C_0^\infty(Q)$ is the space of smooth functions with compact support in Q. Here $\varphi = (\varphi^i)_{i=1}^d$ and

$$b\nabla\varphi = \left(\sum_{j=1}^n \sum_{i=1}^d b_{jk}^i \frac{\partial}{\partial x_j}\varphi^i \right)_{k=1}^d.$$

We now state our main results on interior regularity of weak solutions of (2.1).

THEOREM 2.1. *Assume that $1 \le p, q \le \infty$ satisfies $n/p + 2/q = 1$.*

(i) Suppose that $b \in L^{p,q}(Q_R)$ where Q_R is given in Section 1. Assume that $\omega \in L^{2,2}(Q_R)$ is a weak solution of (2.1) in Q_R. Then there is a positive constant $\varepsilon < 1$ such that $\|b\|_{L^{p,q}(Q_R)} < \varepsilon$ implies

(a) $\omega \in L^{\infty,\beta}(Q_{R/2})$ for all $2 \le \beta < \infty$ when $p > n$.

(b) $\omega \in L^{\alpha,\beta}(Q_{R/2})$ for all $2 \le \alpha, \beta < \infty$ when $p = n$.

Here $\varepsilon = \varepsilon(n,d,p,\beta)$ if $p > n$ and $\varepsilon = \varepsilon(n,d,\alpha,\beta)$ if $p = n$.

(ii) Let $\omega \in L^{2,2}(Q)$ be a weak solution of (2.1) in Q.

(a) If $p > n$ and $b \in L^{p,q}(Q)$, then $\omega \in L^{\infty,\beta}(Q')$ for all $\beta \ge 2$ with $Q' = \Omega' \times (\sigma, T)$, where $\overline{\Omega'}$ is compact in Ω and $\sigma > 0$.

(b) If $b \in L^{n,\infty}(Q)$ and $\|b\|_{L^{n,\infty}(Q)}$ is sufficiently small, then $\omega \in L^{\alpha,\beta}(Q')$ for all $2 \le \alpha, \beta < \infty$.

REMARK: If $n/p + 2/q < 1$, Ladyzenskaya, Ural'ceva and Solonnikov [LUS] showed $\omega \in L^{\infty,\infty}$ under more regularity assumptions on ω than those in Theorem 2.1, where we only need $\omega \in L^{2,2}(Q_R)$ (cf. [LUS] Chap.5, §2).

We recall *Lorentz spaces* $L^{(q)}$ for $1 < q < \infty$:

$$L^{(q)}(0,T) = \{f \in L^1(0,T); [f]_{L^{(q)}(0,T)} < \infty\},$$

where

$$[f]_{L^{(q)}(0,T)} = \sup_{s>0} s(\mu\{t \in (0,T); |f(t)| > s\})^{1/q}.$$

Here μ denotes the Lebesgue measure on $\mathbb{R}$. Although $[f]_{L^{(q)}(0,T)}$ is not a norm (the triangle inequality fails to satisfy), there is an equivalent "norm" in $L^{(q)}(0,T)$ provided that $1 < q < \infty$ and $L^{(q)}(0,T)$ is a Banach space equipped with this norm (cf. [BL]). It thus holds

$$[f+g]_{L^{(q)}(0,T)} \leq C([f]_{L^{(q)}(0,T)} + [g]_{L^{(q)}(0,T)}). \tag{2.3}$$

When $0 < T < \infty$, we see

$$C_\varepsilon [f]_{L^{p-\varepsilon}(0,T)} \leq [f]_{L^{(p)}(0,T)} \leq [f]_{L^p(0,T)} \tag{2.4}$$

for any $\varepsilon > 0$, and that $t^{-1/p} \in L^{(p)}(0,T)$. We now write

$$f(x,t) \in L^{p,(q)}(Q) \qquad \text{if} \quad ||f||_{L^{p,(q)}(Q)} = [\ ||f||_{L^p(\Omega)}(t)\ ]_{L^{(q)}(0,T)} < \infty.$$

THEOREM 2.2. *Assume that $1 \leq p, q \leq \infty$ satisfies $n/p + 2/q = 1$ and $p > n$. Suppose that $\omega \in L^{2,2}(Q_R)$ is a weak solution of (2.1) in Q_R. Then there exists a positive constant $\varepsilon < 1$ such that*

$$||b||_{L^{p,(q)}(Q_R)} < \varepsilon$$

implies

$$\omega \in L^{\infty,\beta}(Q_{R/2}) \quad \textit{for all } \beta > 2.$$

Here $\varepsilon = \varepsilon(n, d, p, \beta)$.

3. Interior Regularity for the Navier-Stokes Equations

As applications of Theorems 2.1 and 2.2, we derive some interior regularity results for weak solutions of the Navier-Stokes equations. Our results extend those of Serrin [Se] and Struwe [St].

We say $u \in L^{2,\infty}(Q)$ with $\nabla u \in L^{2,2}(Q)$ is a *weak solution* of

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla \phi = 0 \\ \nabla \cdot u = 0 \end{cases} \quad \text{in } Q \tag{3.1}$$

if

$$\begin{cases} \displaystyle\iint_Q (\varphi_t + \Delta\varphi + (u \cdot \nabla)\varphi)u \; dxdt = 0 \\ \displaystyle\iint_Q (u \cdot \nabla)\eta \; dxdt = 0, \end{cases} \tag{3.2}$$

for any $\varphi = (\varphi^i)_{i=1}^n \in C_0^\infty(Q)$ with $\nabla \cdot \varphi = 0$ and $\eta \in C_0^\infty(Q)$.

REMARK: If u is a weak solution of (3.1), we see the vorticity $\omega = \text{curl } u$ is a weak solution of (2.1) with $d = n(n-1)/2$ where b_{jk}^i is a linear combination of u^i. For example, if $n = 3$, applying the operator "curl" to (3.1) yields

$$\omega_t - \Delta\omega + \nabla b\omega = 0 \quad \text{with } b_{jk}^i = u^j\delta_{ik} - u^i\delta_{jk}. \tag{3.3}$$

THEOREM 3.1. *If u is a weak solution of (3.1) in Q with*

$$u \in L^{2,\infty}(Q), \; \nabla u \in L^{2,2}(Q) \text{ and}$$

$$\begin{cases} \|u\|_{L^{p,q}(Q)} < \infty \text{ for some } p, q \text{ such that } n/p + 2/q = 1, n < p \le \infty \\ \text{or } \|u\|_{L^{n,\infty}(Q)} \text{ is sufficiently small,} \end{cases}$$

then

$$u \in L^{\infty,\infty}(Q') \text{ and } \operatorname{curl} u \in L^{\infty,\infty}(Q')$$

where Q' is as in Theorem 2.1.

(By Serrin's results in [Se], this theorem yields that u is C^∞ in space variables.)

THEOREM 3.2. *Assume that u is a weak solution of (3.1) in Q_R such that*

$$u \in L^{2,\infty}(Q_R) \text{ and } \nabla u \in L^{2,2}(Q_R).$$

Suppose that $1 \leq p, q \leq \infty$ satisfies $n/p + 2/q = 1$ and $p > n$. Then there exists a positive constant $\varepsilon = \varepsilon(n,p) < 1$ such that

$$\|u\|_{L^{p,(q)}(Q_R)} \leq \varepsilon \tag{3.4}$$

implies

$$u \in L^{\infty,\infty}(Q_{R/4}) \text{ and } \operatorname{curl} u \in L^{\infty,\infty}(Q_{R/4}).$$

REMARK: The condition (3.4) is fulfilled if, for example,

$$\|u(t)\|_{L^p(B_R(x_0))} \leq \frac{\varepsilon}{(t_0 - t)^{1/q}} \quad \text{for} \quad t \in (-R^2 + t_0, t_0).$$

PROOF THAT THEOREM 2.1 IMPLIES THEOREM 3.1: Applying Theorem 2.1(ii) to (3.3) we see $\omega \in L^{\infty,\beta}(Q')$ for any $\beta > 2$. Since $u \in L^{2,\infty}(Q)$ and $-\Delta u = \operatorname{curl}\omega$ in Q, we obtain $u \in L^{\infty,\beta}(Q^2)$ for any $\beta > 2$ by a standard argument (cf. Serrin [Se], P193, Step II). As in Serrin [Se], the remark to Theorem 2.1 yields $\omega \in L^{\infty,\infty}(Q^3)$, which implies $u \in L^{\infty,\infty}(Q^4)$. Here $Q^i = \Omega^i \times (\sigma_i, T)$, $\Omega^{i+1} \Subset \Omega^i$, $\sigma_{i+1} > \sigma_i$ for $1 \leq i \leq 4$ and $Q^1 = Q'$. ∎

PROOF THAT THEOREM 2.2 IMPLIES THEOREM 3.2: If ε is sufficiently small, applying Theorem 2.2 with $\omega := \operatorname{curl} u$ yields

$$\omega \in L^{\infty,\beta}(Q_{R/2}) \quad \text{for any } \beta > 2.$$

The proof of Theorem 3.1 now yields

$$u \in L^{\infty,\infty}(Q_{R/4}) \text{ and } \operatorname{curl} u \in L^{\infty,\infty}(Q_{R/4}). \quad ∎$$

Part 2

ON A REGULARITY CRITERION UP TO THE BOUNDARY FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

Abstract. We are concerned with the behavior of weak solutions of the Navier-Stokes system around possible singularities on the boundary. We show that a weak solution locally belonging to some Lebesgue space can not blowup.

1. Introduction

We consider the Navier-Stokes equations:

$$(1.1)\qquad \begin{cases} u_t - \Delta u + (u\cdot\nabla)u + \nabla\phi = 0, & \text{in } Q = \Omega\times(-T,0),\\ \nabla\cdot u = 0, & \text{in } Q,\\ u(x,-T) = u_0(x), & \text{on } \Omega,\\ u|_{\partial\Omega} = 0, & \end{cases}$$

where Ω is a domain in $\mathbb{R}^n (n \geqslant 3)$ with smooth boundary $\partial\Omega$, $0 < T < \infty$; $u = (u^i)_{i=1}^n$ and ϕ denote the unknown velocity and pressure, respectively, while $u_0 = (u_0^i)_{i=1}^n$ is a given initial velocity. Here external force is assumed to be zero for simplicity. Leray [Le] and Hopf [Ho] constructed global weak solutions in the class

$$(1.2)\qquad u \in L^{2,\infty}(Q) \quad \text{and} \quad \nabla u \in L^{2,2}(Q)$$

for $u_0 \in L^2(\Omega)$ where $L^{p,q}(Q) = L^q(-T,0;L^p(\Omega))$. It is also known that there exist weak solutions moreover in the class

$$(1.3)\qquad \nabla u, \phi \in L^{r_0,r_0'}(Q)$$

for all $1 < r_0, r_0' < \infty$ such that $n/r_0 + 2/r_0' = n$ for some smooth initial data (cf. Giga and Sohr [GS], Sohr and von Wahl [SW]). Serrin [Se] gave a local interior regularity criterion and Struwe [St] extended Serrin's result (cf. Takahashi [Ta]). They proved that the weak solution u in the class (1.2) is in $L^{\infty,\infty}(Q')$ and regular in the space variables provided that $u \in L^{p,q}(Q)$ for some p, q such that

$$n/p + 2/q \leqslant 1, \quad n < p \leqslant \infty, \tag{1.4}$$

where $Q' = \Omega' \times (-T', 0)$, Ω' is relatively compact in Ω and $T' < T$. When $\Omega = \mathbb{R}^n$, this was proved by Fabes, Jones and Riviere [FJR] (See also von Wahl [Wa]).

Although global versions of Serrin-Struwe's results are available (cf. Giga [Gi], Sohr [So]), there seems no literature on a local version *up to the boundary*. Our goal is to give a local regularity criterion *up to the boundary* of Serrin-Struwe type. For simplicity we first assume that the boundary $\partial\Omega$ is flat near a possible blowup point $x_0 \in \partial\Omega$. By changing variables we may assume that $x_0 = 0$. We take R so small that $\partial\Omega \cap B_R(0)$ is flat. Here $B_R(0)$ denotes the ball centered at 0 with redius R. We prove among other results in this paper that the weak solution u in the class (1.2) and (1.3) satisfying $u \in L^{p,q}(Q \cap Q_R)$ with

$$n/p + 2/q = 1, \quad n < p \leqslant \infty \tag{1.5}$$

implies

$$u \in L^{\infty,\infty}(Q \cap Q_{R'}),$$

where $Q_R = B_R(0) \times (-R^2, 0)$, $R^2 \leqslant T$ and $R' < R$. However, we are not sure whether the boundedness of u in space-time would imply the smoothness of u up to the boundary in the space variables, while it is true on the interior probrem (cf. [Se]). Concerning the interior regularity problem, the vorticity equation has been fully used (cf. Serrin [Se], Struwe [St] and Takahashi [Ta]). In our case, such a equation is not available, because we can not specify the boundary value of the vorticity $\omega = \operatorname{curl} u$ locally. Hence we here analize (1.1) directly. When we localize the velocity, there arises also such a problem that the localized velocity is no longer solenoidal. We recover this difficulty with a variant of Bogovski's lemma which gives a solution of $\nabla \cdot v = f$ with zero boundary condition (cf. Bogovski [Bo1],[Bo2] and Borchers and Sohr [BS]).

2. Main theorem

We denote $Q_R^+ = B_R^+ \times (-R^2, 0)$, $B_R^+ = \{x \in \mathbb{R}^n \mid |x| < R, x_n > 0\}$ and $L^{p,q}(Q_R^+) = L^q(-R^2, 0; L^p(B_R^+))$.

We say (u, ϕ) in the class

$$\text{(2.1)} \qquad \begin{cases} u \in L^{2,\infty}(Q_R^+), \quad \nabla u \in L^{2,2}(Q_R^+), \\ u(\cdot, t)|_{x_n=0} = 0 \quad \text{for almost every } t \in (-R^2, 0), \end{cases}$$

is a *weak solution* of

$$\text{(2.2)} \qquad \begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla\phi = 0, \\ \nabla \cdot u = 0, \\ u|_{x_n=0} = 0, \end{cases} \qquad \text{in } Q_R^+,$$

if it holds

$$\text{(2.3)} \qquad \begin{cases} \displaystyle\iint_{Q_R^+} \{(\varphi_t + \Delta\varphi + (u \cdot \nabla)\varphi) \cdot u - (\phi\nabla \cdot \varphi)\}\, dx\, dt = 0, \\ \displaystyle\iint_{Q_R^+} (u \cdot \nabla)\eta\, dx\, dt = 0 \end{cases}$$

for all $\varphi = (\varphi^i)_{i=1}^n \in C_0^\infty(Q_R^+)$, and for all $\eta \in C_0^\infty(Q_R^+)$. Here $C_0^\infty(Q)$ is the space of smooth functions with compact support in Q.

We do not distinguish the spaces of vector and scalar valued functions unless it causes confusion. We now state our main result.

THEOREM 2.1. *Suppose that (u, ϕ) is a weak solution of (2.2) in the class (2.1) and*

$$\text{(2.4)} \qquad \nabla u,\ \phi \in L^{r_0, r_0'}(Q_R^+) \quad \text{for all } 1 < r_0, r_0' < \infty \text{ with } \frac{n}{r_0} + \frac{2}{r_0'} = n.$$

(a) *Assume that $1 \leqslant p, q \leqslant \infty$ satisfies $n/p + 2/q = 1$ and $p > n$. If $u \in L^{p,q}(Q_R^+)$, then*

$$u \in L^{\infty,\infty}(Q_{R/8}^+),$$
$$\nabla u,\ \phi \in L^{\alpha,\alpha'}(Q_{R/4}^+) \quad \text{for all } 2 \leqslant \alpha, \alpha' < \infty.$$

(b) *There exists a positive constant $\varepsilon = \varepsilon(n) < 1$ such that $\|u\|_{L^{n,\infty}(Q_R^+)} < \varepsilon$ implies that*

$$u \in L^{\infty,\infty}(Q_{R/8}^+),$$
$$\nabla u,\ \phi \in L^{\alpha,\alpha'}(Q_{R/4}^+) \quad \text{for all } 2 \leqslant \alpha, \alpha' < \infty.$$

3. Localization

We denote $\widetilde{B_R^+} = \{x \in \mathbb{R}^n \big| \, |x| < R, x_n \geqslant 0\}$. We first assume that $R = 1$. We cut off a weak solution (u, ϕ) of (2.2) on $Q_{1/2}^+$ to obtain higher regularity in $Q_{1/2}^+$. We set

$$\widetilde{u} = u\psi \quad \text{and} \quad \rho = \phi\psi$$

where $\psi \in C_0^\infty(\widetilde{B_1^+} \times (-1, 0])$ satisfies

$$\psi = 1 \qquad \text{in} \quad \overline{B_{1/2}^+} \times (-1/4, 0].$$

Then $(\widetilde{u}, \rho)$ satisfies

$$(3.1) \qquad \begin{cases} \widetilde{u}_t - \Delta\widetilde{u} + (u \cdot \nabla)\widetilde{u} + \nabla\rho = \phi\nabla\psi + \zeta(u, \psi), & \text{in } Q_1^+, \\ \nabla \cdot \widetilde{u} = u \cdot \nabla\psi, & \text{in } Q_1^+, \\ \widetilde{u}(x, -1) = 0, & \text{on } B_1^+, \\ \widetilde{u}\big|_{x_n = 0} = 0, \end{cases}$$

where

$$\zeta(u, \psi) = \psi_t u + u\Delta\psi - 2\nabla(u\nabla\psi) + (u \cdot \nabla\psi)u.$$

However $\widetilde{u}$ may not satisfy the incompressibility condition $\nabla \cdot \widetilde{u} = 0$. We recover this condition with a variant of Bogovski's lemma. To state it we prepare some function spaces:

Let D be a bounded domain in $\mathbb{R}^n$. Let $H^{j,r}(D)$ be the completion of $C^\infty(\overline{D})$ with respect to the norm $|\cdot|_{j,r}$, where $|f|_{j,r}^r = \sum_{|\alpha| \leqslant j} ||\nabla^\alpha f||_r^r$. Here we denote

$$\nabla^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $||f||_r^r = \int_D |f|^r dx$. $H_\Gamma^{j,r}(D)$ is the completion of $C_0^\infty(D \cup \Gamma)$ with respect to $|\cdot|_{j,r}$ where Γ is a closed set on ∂D. We denote the support of f by $\operatorname{supp} f$ and denote $H_\Gamma^{j,r}(D)$ by $H_0^{j,r}(D)$ if Γ is empty. We write $\nabla_i = \dfrac{\partial}{\partial x_i}$.

REMARK: $H^{j,r}(D)$ coincides with the usual Sobolev space $W^{j,r}(D)$ for such a wider class of domains D as have Lipschitz continuous boundaries. (See [GT, Section 7.6] and [Ad, 3.18]).

LEMMA 3.1. *Assume that D is a bounded Lipschitz domain in $\mathbb{R}^n$, Γ is a closed subset on ∂D with smooth boundary $\partial\Gamma$ and ∂D is smooth on Γ. For any $j = 0, 1, 2, \cdots$, and any $r \in (1, \infty)$, there exist a bounded linear operator $K = K_{j,r} : H_\Gamma^{j,r}(D) \longrightarrow H_\Gamma^{j+1,r}(D)^n \cap H_0^{1,r}(D)^n$ and positive constants $C = C(n, j, r, D)$ and $C' = C'(n, r, D)$ with the following properties:*

(a) $\nabla \cdot Kf = f$ *for all* $f \in H_\Gamma^{j,r}(D)$ *with* $\int_D f\,dx = 0$,

(b) $\|\nabla^{j+1} Kf\|_r \leqslant C|f|_{j,r}$ *for all* $f \in H_\Gamma^{j,r}(D)$,

(c) *if the* $n-1$ *dimensional Hausdorff measure of* $\partial D\backslash\Gamma$ *is positive,*

$$\|\nabla^{j+1} Kf\|_r \leqslant C\|\nabla^j f\|_r \quad \text{for all } f \in H_\Gamma^{j,r}(D),$$

(d) $\operatorname{supp} Kf \subset D \cup \Gamma$ *if* $\operatorname{supp} f \subset D \cup \Gamma$,

(e) *for* $f \in L^r(D)$, *we can define* $K(\nabla_i f) \in L^r(D)$ $(i = 1, \cdots, n)$ *such that* $\nabla \cdot K(\nabla_i f) = \nabla_i f$ *for* $f \in H^{1,r}(D)$ *and that*

$$\|K(\nabla_i f)\|_r \leqslant C'\|f\|_r \quad \text{for all } f \in L^r(D).$$

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