数理解析研究所講究録
第 745 巻 1991 年 102-109

Example of zero viscosity limit for two dimensional nonstationary Navier–Stokes flows with boundary

北海道情報大学　松井伸也　(Shin'ya MATSUI)

1. Introduction

Our purpose in this report is to give an example for a flow u^ν of the nonstationary, incompressible Navier–Stokes equations in Ω which is convergent to an Euler flow $\overline{u}$ as the viscosity ν tends to zero, where Ω is a bounded domain with smooth boundary.

Let $(u^\nu(t), p^\nu(t))$ be the unique global classical solution of the Navier–Stokes equations for the viscosity ν (we omit a variable $x \in \Omega$ for simplicity):

$$
\text{(NS)} \qquad \begin{aligned} &u^\nu_t - \nu\Delta u^\nu + (u^\nu, \nabla)u^\nu + \nabla p^\nu = f^\nu, \\ &\operatorname{div} u^\nu = 0 \quad \text{in } \Omega \times (0,T), \\ &u^\nu|_{\partial\Omega} = 0, \ u^\nu|_{t=0} = u^\nu_0, \end{aligned}
$$

where $f^\nu(t)$ and u^ν_0 are outer forces and initial data which satisfy the compatibility conditions $u^\nu_0|_{\partial\Omega} = 0$ and $\operatorname{div} u^\nu_0 = 0$ (for existence, see [3]). If $\nu \to 0$ in (NS) formally, we have the Euler equations which has the unique global classical solution $(\overline{u}(t), \overline{p}(t))$ (see, [1]):

$$
\text{(EE)} \qquad \begin{aligned} &\overline{u}_t + (\overline{u}, \nabla)\overline{u} + \nabla\overline{p} = \overline{f}, \\ &\operatorname{div} \overline{u} = 0 \quad \text{in } \Omega \times (0,T), \\ &\overline{u} \cdot n|_{\partial\Omega} = 0, \ \overline{u}|_{t=0} = \overline{u}_0, \end{aligned}
$$

where $\overline{f}(t), \overline{u}_0$ and n are outer forces, initial data and the unit outer normal to $\partial\Omega$ respectively with $\overline{u}_0$ satisfying the compatibility conditions $\overline{u}_0 \cdot n|_{\partial\Omega} = 0$ and $\operatorname{div} \overline{u}_0 = 0$.

To prove the convergent of our flow in the example we need

Theorem 1. *Assume*

$$
(1) \qquad u^\nu_0 \to \overline{u}_0 \quad \text{as } \nu \to 0 \text{ in } L^2(\Omega),
$$

$$
(2) \qquad f^\nu \to \overline{f} \quad \text{as } \nu \to 0 \text{ in } L^1(0,T; L^2(\Omega)).
$$

Then the following three conditions are equivalent for $t \in [0,T]$.

(a) $||u^\nu(t) - \overline{u}(t)||_{L^2(\Omega)} \to 0$ *as* $\nu \to 0$ *uniformly (pointwisely) in* t,

(b) $\overline{\lim_{\nu\to 0}}\, \nu \int_0^t \int_{\partial\Omega} \overline{u}(\tau)\cdot n \times \operatorname{rot} u^\nu(\tau)\, dSd\tau = 0$ *uniformly (pointwisely) in* t,

(c) $\lim_{\nu\to 0} \nu \int_0^t \int_{\partial\Omega} \overline{u}(\tau)\cdot n \times \operatorname{rot} u^\nu(\tau)\, dSd\tau = 0$ *uniformly (pointwisely) in* t,

where dS denotes surface area of $\partial\Omega$, $\operatorname{rot} u = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ for vector fields $u(x) = (u_1(x), u_2(x))$ in $x = (x_1, x_2)$ and $a \times b = (a_2 b, -a_1 b)$ for a vector $a = (a_1, a_2)$ and a scholar b.

Remark. (1) Shirota also obtained Theorem 1 in somewhat different statements independently of ours, which is not published.

(2) Kato[2] obtained other equivalent conditions to (a) for the flows in a bounded domain of $\mathbf{R}^n$. One of them is

$$\nu \int_0^T ||\operatorname{grad} u^\nu(\tau)||^2_{L^2(\Gamma_{c\nu})}\, d\tau \to 0 \text{ as } \nu \to 0,$$

where $\Gamma_{c\nu}$ is the boundary strip of width $c\nu$ with $c > 0$ fixed.

2. Example

In this section Ω is the open unit disk $\{x = (x_1, x_2) \in \mathbf{R}^2; |x| = (x_1^2 + x_2^2)^{1/2} < 1\}$. For simplicity we denote $r = |x|$ and ${}^t(\cos\theta, \sin\theta) = x/|x|$, where ${}^t(\cdot,\cdot)$ is a transported vector of $(\cdot,\cdot)$. We note that the unit outer normal to $\partial\Omega$ is $x/|x|$. Furthermore we assume $f^\nu = \overline{f} = 0$.

We employ the stationary solution $\overline{u}$, defined by a rotating eddy, to the Euler equations (see, [4]):

$$\overline{u}(x)\ (= \overline{u}_0(x)) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \frac{1}{r} \int_0^r \rho\overline{\omega}_0(\rho)\, d\rho. \tag{2.1}$$

For any function $\overline{\omega}_0 \in C([0,1])$ we have

$$\operatorname{div} \overline{u} = 0 \quad \text{in } \overline{\Omega}, \tag{2.2a}$$

$$\overline{u} \cdot n = 0 \quad \text{on } \partial\Omega, \tag{2.2b}$$

$$\operatorname{rot} \overline{u} = \overline{\omega}_0 \quad \text{in } \overline{\Omega}, \tag{2.2c}$$

$$(\overline{u}, \nabla)\overline{u} = -\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \frac{\overline{\varphi}^2}{r^3} = \nabla \overline{F} \quad \text{in } \overline{\Omega}, \tag{2.2d}$$

where $\overline{\varphi}(r) = \int_0^r \rho\overline{\omega}_0(\rho)\, d\rho$ and $\overline{F}(r) = -\int_0^r \overline{\varphi}^2(s)/s^3\, ds$ which is well defined in $[0,1]$, since

$$|\overline{\varphi}(s)|^2 \leq \int_0^s \rho^2\, d\rho \cdot \int_0^s \overline{\omega}^2(\rho)\, d\rho \leq \frac{1}{3}s^3 \|\overline{\omega}\|^2_{L^2(0,1)}. \tag{2.3}$$

Thus, $(\overline{u}, \overline{p})$ is the solution of (EE) for $\overline{f} = 0$, if $\overline{u}$ is in (2.1) and $\nabla\overline{p} = -\nabla\overline{F}$.

We construct a non–stationary solution of (NS) in the form:

$$u^\nu(x,t) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \frac{1}{r} \int_0^r \rho\omega^\nu(\rho,t)\, d\rho, \tag{2.4}$$

where $\omega^\nu(r,t)$ is unknown. We note that $u^\nu(x,t)$ in (2.4) satisfies the same identities as (2.2a)–(2.2d).

To construct $u^\nu(t)$, we reduce (NS) to an equation of

$$\varphi^\nu(r,t) = \int_0^r \rho\omega^\nu(\rho,t)\, d\rho \tag{2.5}$$

instead of $\omega^\nu = \operatorname{rot} u^\nu$. By (2.2d) we have

$$\begin{aligned} &u_t^\nu - \nu\Delta u^\nu + (u^\nu, \nabla)u^\nu + \nabla p^\nu \\ &= \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \frac{1}{r}(\varphi_t^\nu - \nu\varphi_{rr}^\nu + \frac{\nu}{r}\varphi_r^\nu) + \nabla(F^\nu + p^\nu) = 0, \end{aligned}$$

where $F^\nu = -\int_0^r (\varphi^\nu)^2(s,t)/s^3\, ds$ which is well defined because of (2.3), if $\omega^\nu \in L^\infty(0,T; L^2(\Omega))$. Since a vector field ${}^t(-\sin\theta, \cos\theta)\Phi(r)$ is solenoidal for a radially symmetric function $\Phi(r)$, that is,

$$\operatorname{div}\{\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \Phi(r)\} = 0 \text{ in } \Omega \text{ and } \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \Phi(r) \cdot n = 0 \text{ on } \partial\Omega,$$

then the equation of φ^ν is

$$\text{(E)}\qquad \begin{aligned} &\varphi^\nu_t - \nu\varphi^\nu_{rr} + \frac{\nu}{r}\varphi^\nu_r = 0 \quad \text{for } (r,t) \in Q_T \equiv (0,1)\times(0,T),\\ &\varphi^\nu_r|_{r=0} = 0,\ \varphi^\nu|_{r=1} = 0 \quad \text{for } t \in (0,T),\\ &\varphi^\nu|_{t=0} = \varphi^\nu_0 \equiv \int_0^r \rho\omega^\nu_0(\rho)\,d\rho \quad \text{for } r \in (0,1), \end{aligned}$$

here $\omega^\nu_0 = \operatorname{rot} u^\nu_0$ is given data, T is any but fixed positive number and a subscript of φ^ν denotes partial differential with respect to its variable.

Thus let $\varphi^\nu(t)$ be a solution of (E). Then (u^ν, p^ν) is a solution of (NS) for $f^\nu = 0$ and $u^\nu_0 = {}^t(-\sin\theta, \cos\theta)\varphi^\nu_0/r$, if $u^\nu(t)$ is defined by (2.4) and $p^\nu(t)$ is a solution of

$$\begin{aligned} \Delta p^\nu &= -\Delta F^\nu \quad \text{in } \Omega,\\ \nabla p^\nu \cdot n &= -\nabla F^\nu \cdot n \quad \text{on } \partial\Omega. \end{aligned}$$

For existence of a solution to (E) we have

Theorem 2 (Existence of the flow). *Assume*

$$\omega^\nu_0(r) = \frac{1}{r}\partial_r(\varphi^\nu_0)(r), \ \textit{that is,}\ \varphi^\nu_0(r) = \int_0^r \rho\omega^\nu_0(\rho)\,d\rho$$

for $\varphi^\nu_0 \in C^{2+\alpha}([0,1])$ *with* $\varphi^\nu_0(0) = \partial_r(\varphi^\nu_0)(0) = 0$ *and* $0 < \alpha < 1$. *Then there exists an unique solution* $\varphi^\nu \in C^{2,1}(Q)$ *of (E), which satisfies*

$$\begin{aligned} &\varphi^\nu(0,t) = 0 \quad \textit{for} \quad 0 \le t < \infty,\\ &|\varphi^\nu(r,t)| \le \frac{\sqrt{3}}{3}\|\omega^\nu_0\|_{L^2(0,1)} \quad \textit{in} \quad Q, \end{aligned}$$

where $Q = \{(r,t); 0 \le r \le 1,\ 0 \le t < \infty \textit{ and } (r,t) \ne (1,0)\}$ *and* ∂_r *denotes the differential operator* $\frac{d}{dr}$.

Here $C^{2,1}(Q)$ (resp. $C^{2+\alpha}([0,1])$) is the Banach space whose elements have second derivatives in r and first derivatives in t (resp. second derivatives in r). Furthermore second derivatives of the elements in $C^{2+\alpha}([0,1])$ are Hölder continuous with exponent α in $r \in [0,1]$.

Remark. In Theorem 2 we don't require the compatibility condition $\varphi_0^\nu(1) = 0$. Thus for the existence of a solution $u^\nu(t)$ in (2.4) to (NS) we don't need to assume $u_0^\nu|_{\partial\Omega} = 0$. Hence our solution $u^\nu(t)$ has the initial layer.

Finally our example is

THEOREM 3 (CONVERGENCE OF THE FLOW). *Assume the same in Theorem 2 and* $\overline{\omega}_0 \in C([0,1])$ *in (2.1). We put* $u_0^\nu = {}^t(-\sin\theta, \cos\theta)\varphi_0^\nu/r$ *and let* $\overline{u}$ *and* $u^\nu(t)$ *be in (2.1) and (2.4) respectively. Finally we assume that* $u_0^\nu \to \overline{u}_0$ *in* $L^2(\Omega)$ *as* $\nu \to 0$ *and* $||\omega_0^\nu||_{L^2(0,1)} \le C$ *independent of the viscosity* ν*. Then we obtain for any but fixed* $T > 0$

$$||u^\nu(t) - \overline{u}||_{L^2(\Omega)} \to 0 \quad \text{as } \nu \to 0 \text{ uniformly in } t \in [0, T].$$

Remark. (1) Since we don't require $\varphi_0^\nu(1) = 0$, we can take $\overline{u}_0$ as the initial data of (NS). (2) If we assume that the compatibility condition $\varphi_0^\nu(1) = 0$ in Theorem 2, then by arguments likely to the below we can obtain

$$u^\nu \to \overline{u} \quad \text{in } C(K) \quad \text{as } \nu \to 0$$

for any compact subset $K \subset \overline{Q}$, even if $|\omega_0^\nu(r)| \le \nu^{-\varepsilon}$ for $1 - \nu^{2\varepsilon} \le r \le 1$ and $\varepsilon < 1$ fixed. The proof is omitted in this report.

The remaining part in this section is to prove Theorem 3.

We denote by $\psi(r,t)$, the solution of

$$\text{(E')} \qquad \begin{aligned} &\psi_t - \psi_{rr} + \frac{1}{r}\psi_r = 0 \quad \text{for } (r,t) \in Q_T, \\ &\psi_r|_{r=0} = 0, \ \psi|_{r=1} = 0 \quad \text{for } t \in (0,T), \\ &\psi|_{t=0} = \varphi_0^\nu \equiv \int_0^r \rho\omega_0^\nu(\rho)\, d\rho \quad \text{for } r \in (0,1). \end{aligned}$$

Then the uniqueness of the solution to (E) implies

LEMMA 1. *Let* $\varphi^\nu(t)$ *be the solution of (E) in Theorem 2. Then we obtain*

$$\varphi^\nu(r,t) = \psi(r, \nu t) \quad \text{in } Q$$

for a fixed ν.

The following lemma plays the essential role in the proof of Theorem 3.

LEMMA 2. *Let $\varphi^\nu(t)$ be the solution in Theorem 2. Then*

$$|\int_0^t \varphi_r^\nu(1,\tau)\,d\tau| \le C(\|\omega_0^\nu\|_{L^2(0,1)}+1)\exp C(\|\omega_0^\nu\|_{L^2(0,1)}T+1)$$

for any $t \in [0,T]$, where C denotes several different positive constants independently of ν and T here and after.

Proof. Let $\psi(t)$ be in Lemma 1 and ν be fixed. In (E') we replace t by νt. Then it follows that

$$\psi_{rr}(r,\nu t) - \frac{1}{r}\psi_r(r,\nu t) - \psi_t(r,\nu t) = 0 \quad \text{in } Q.$$

To integrate this equation in t on (ε, t) for any but fixed $\varepsilon > 0$, then $f(r,t) = \int_\varepsilon^t \psi(r,\nu\tau)\,d\tau$ satisfies

$$f_{rr} - \frac{1}{r}f_r - f_t = a_\varepsilon \quad \text{in } Q_T^\varepsilon = (0,1)\times(\varepsilon,T),$$
$$f_r|_{r=0} = 0,\ f|_{r=1} = 0 \quad \text{for } t \in (\varepsilon,T),$$
$$f|_{t=\varepsilon} = 0 \quad \text{for } r \in (0,1),$$

where $a_\varepsilon(r) = \psi(r,\nu\varepsilon)$.

For $\chi \in C^\infty(\mathbf{R})$ which satisfies $0 \le \chi(r) \le 1$, $\chi = 1$ in $[2/3,\infty)$ and $\chi = 0$ in $(-\infty, 1/3]$, we put $z(t) = \chi^2 \exp f(t)$ and $Pz = z_{rr} - z_t$. Then we have

$$\begin{aligned} Pz &= (\chi^2)''e^f + 4\chi\chi' f_r e^f + \chi^2 f_r^2 e^f + \chi^2 f_{rr} e^f - \chi^2 f_t e^f \\ &= e^f\{\chi^2(f_{rr} - f_t) + (\chi^2)'' + 4\chi\chi' f_r + \chi^2 f_r^2\} \\ &= e^f\{\frac{\chi^2}{r}f_r + \chi^2 a_\varepsilon + (\chi^2)'' + 4\chi\chi' f_r + \chi^2 f_r^2\}. \end{aligned}$$

Since absolute values of χ^2/r, χ' and $(\chi^2)''$ are estimated by C for $r \in [0,1]$, we obtain

$$Pz \ge Ce^f\{-\frac{1}{\mu} - \mu\chi^2 f_r^2 - \chi^2|a_\varepsilon| - 1 - \frac{1}{\mu} - \mu\chi^2 f_r^2 + \chi^2 f_r^2\}$$

for any $\mu > 0$.

Using the estimate $|\psi(r,\nu t)| \leq (1/\sqrt{3})||\omega_0||_{L^2(0,1)}$ for any $(r,t) \in Q$ in Theorem 2 and taking $\mu = 1/2$, then

$$\begin{aligned} Pz &\geq -C(||\omega_0^\nu||_{L^2(0,1)} + 1)\exp(C||\omega_0^\nu||_{L^2(0,1)}T) \\ &\equiv -M_1 e^{M_2}. \end{aligned}$$

Putting $y = z + 2M_1\exp(M_2 + r)$, then $Py > 0$ holds. Hence the maximum principle implies $y(t)$ does not take its maximum in $Q_T^\varepsilon \equiv [0,1] \times [\varepsilon, T]$ at $(r,t) \in (0,1) \times (\varepsilon, T]$. On the other hand, at parabolic boundary of Q_T^ε, $y(t)$ holds as follows:

$$\begin{aligned} y|_{r=0} &= z|_{r=0} + 2M_1 e^{M_2} = 2M_1 e^{M_2}, \\ y|_{t=\varepsilon} &= z|_{t=\varepsilon} + 2M_1 e^{M_2+r} = 2M_1 e^{M_2+r}, \\ y|_{r=1} &= z|_{r=1} + 2M_1 e^{M_2+1} = 2M_1 e^{M_2+1}, \end{aligned}$$

Hence at the all points $(1,t)$ with $\varepsilon \leq t \leq T$, $y(r,t)$ attains its maximum in Q_T^ε. Then we conclude that

$$\frac{\partial y}{\partial r}|_{r=1} = \int_\varepsilon^t \psi_r(r,\nu\tau)\, d\tau + 2M_1 e^{M_2+1} \geq 0.$$

Putting $\varepsilon \to 0$, then

$$\int_0^t \psi_r(r,\nu\tau)\, d\tau \geq -2M_1 e^{M_2+1}.$$

The estimate from above of $f(r,t)$ with $\varepsilon = 0$ can be established in a similar way by making the substitution $\hat{z} = \chi^2\exp(-f)$ and considering $\hat{y} = \hat{z} - 2M_1\exp(M_2 + r)$.

Hence by the identity in Lemma 1 the proof of our estimate is completed.

Finally we note that in this proof we use the method of the proof to Lemma 3 of Section 3 in [6]. □

Now we show Theorem 3. Since

$$\begin{aligned} \operatorname{rot} u^\nu|_{\partial\Omega} = \omega^\nu(r,t)|_{r=1} &= \frac{\partial}{\partial r}\int_0^r \rho\omega^\nu(\rho,t)\, d\rho|_{r=1} \\ &= \varphi_r^\nu(1,t), \end{aligned}$$

we have

$$\overline{u} \cdot n \times \operatorname{rot} u^{\nu}(t)|_{\partial\Omega} = -\varphi_r^{\nu}(1,t) \int_0^1 \rho \overline{\omega}_0(\rho)\, d\rho.$$

Thus we obtain an identity

$$\nu \int_0^t \int_{\partial\Omega} \overline{u} \cdot n \times \operatorname{rot} u^{\nu}(\tau)\, dS d\tau = -2\pi\nu \int_0^1 \rho \overline{\omega}_0(\rho)\, d\rho \cdot \int_0^t \varphi_r^{\nu}(1,\tau)\, d\tau.$$

Hence it is easy to show that (c) in Theorem 1 holds because of Lemma 2. This proves Theorem 3 by Theorem 1.

For the proofs of Theorem 1 and Theorem 2, see [5].

References

[1] Kato, T.: *On classical solution for the two-dimensional non-stationary Euler equation*, Arch. Rat. Mech. Anal. **25** (1967), 188–200.

[2] Kato, T.: *Remarks on zero viscosity limit for nonstationary Navier–Stokes flows with boundary*, in "Seminar on nonlinear partial differential equation, ed. by S. S. Chern," Springer, 1982, pp. 85–98.

[3] Kato, T. and Fujiwara, H.: *On the nonstationary Navier–Stokes system*, Rend. Semi. Math. Univ. Padova **32** (1962), 243–260.

[4] Majida, A.: *Vorticity and the mathematical theory of incompressible fluid flow*, Comm. Pure and Appl. Math. **39** (1986), 187–220.

[5] Matsui, S.: *Example of zero viscosity limit for two dimensional nonstationary Navier–Stokes flows with boundary.* to appear.

[6] Oleinik, O. A. and Kruzhkov, S. N.: *Quasi-linear second order parabolic equations with many independent variables*, Russian Math. Surv. **16** (1961), 106–146.