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Periodic solutions of Boussinesq equations

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Let Ω be a bounded domain in R^2 with the boundary $\partial\Omega$ such that

$$\partial\Omega=\Gamma_1\cup\Gamma_2,\ \Gamma_1\cap\Gamma_2=\phi.$$

We consider the following initial boundary value problem:

$$\begin{cases}\dfrac{\partial u}{\partial t}+(u\cdot\nabla)u=-\dfrac{1}{\rho}\nabla p+\nu\Delta u+\beta g\theta & \\ \mathrm{div}u=0, & x\in\Omega, t>0, \\ \dfrac{\partial\theta}{\partial t}+(u\cdot\nabla)\theta=\chi\Delta\theta, & \end{cases}\tag{1}$$

$$\begin{cases}u(x,t)=0,\ \theta(x,t)=\xi(x,t), & x\in\Gamma_1, t>0,\\ u(x,t)=0, \dfrac{\partial}{\partial n}\theta(x,t)=\eta(x,t), & x\in\Gamma_2, t>0,\end{cases}\tag{2}$$

$$\begin{cases}u(x,0)=a_0(x), \\ \theta(x,0)=\tau_0(x),\end{cases}\quad x\in\Omega,\tag{3}$$

where $u=(u_1,u_2)$ is the fluid velocity, p is the pressure, θ is the temperature, $u\cdot\nabla=\sum_{j=1}^{2}u_j\dfrac{\partial}{\partial x_j}$, $\dfrac{\partial\theta}{\partial n}$ denotes the outer normal derivative of θ at x to $\partial\Omega$, $g(x,t)$ is the gravitational vector function, and ρ(density), ν(kinematic viscosity), β(coefficient of volume expansion), χ(thermal diffusivity) are positive constants. $\xi(x,t)$ (resp. $\eta(x,t)$) is a function defined on $\Gamma_1\times(0,T)$(resp. $\Gamma_2\times(0,T)$) and $a_0(x)$(resp. $\tau_0(x)$) is a vector (resp. scalar) function defined on Ω.

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In order to state our results, we introduce some

Function spaces([1],[2],[3]).

$L^p(\Omega)$ and the Sobolev space $W^\ell_p(\Omega)$ are defined as usual. We also denote $\mathbf{L}^p(\Omega) = L^p(\Omega) \times L^p(\Omega)$, $H^\ell(\Omega) = W^\ell_2(\Omega)$. Whether the elements of the space are scalar or vector functions is understood from the contexts unless stated explicitly.

$D_\sigma = \{$vector function $\varphi \in C^\infty(\Omega) \mid \mathrm{supp}\varphi \subset \Omega, \mathrm{div}\varphi = 0$ in $\Omega\}$,

$H =$ completion of D_σ under the $L^2(\Omega)$-norm,

$V =$ completion of D_σ under the $H^1(\Omega)$-norm,

$D_0 = \{$scalar function $\varphi \in C^\infty(\overline{\Omega}) \mid \varphi \equiv 0$ in a neighborhood of $\Gamma_1\}$,

$W =$ completion of D_0 under the $H^1(\Omega)$-norm,

V', W' are dual space of V, W.

Definition 1

$\{u, \theta\}$ is called a weak solution of evolutional problem (1),(2) if, for some function θ_0 such that

$$\theta_0 \in L^2(0, T : H^1(\Omega)), \quad \theta_0 = \xi \text{ on } \Gamma_1,$$

$\{u, \theta\}$ satisfies following conditions:

$u \in L^2(0, T : V), \quad \theta - \theta_0 \in L^2(0, T : W)$,

$$\begin{cases} \dfrac{d}{dt}(u, v) + \nu(\nabla u, \nabla v) + ((u \cdot \nabla)u, v) - (\beta g \theta, v) = 0, & \forall v \in V, \\ \dfrac{d}{dt}(\theta, \tau) + \chi(\nabla \theta, \nabla \tau) + ((u \cdot \nabla)\theta, \tau) - \chi(\eta, \tau)_{\Gamma_2} = 0, & \forall \tau \in W, \end{cases} \tag{4}$$

where

$$(\eta, \tau)_{\Gamma_2} = \int_{\Gamma_2} \eta(x')\tau(x')d\sigma.$$

As for the smoothness of $\partial\Omega$, we suppose

Condition (H)

$\partial\Omega$ is of class C^1 and devided as follows:

$$\partial\Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \phi, \qquad \text{measure of } \Gamma_1 \neq 0,$$

and the intersection $\overline{\Gamma}_1 \cap \overline{\Gamma}_2$ consists of finite points.

In [3], we showed the existence and the uniqueness of weak solution of evolutional problem for $2 \leq n \leq 4$. For $n = 2$, we have the following result:

Theorem A

Let Ω be a bounded domain in R^2 with C^1 boundary satisfying Condition(H). If the function g is in $L^\infty(\Omega \times (0,T))$, $\xi \in C^1(\overline{\Gamma}_1 \times [0,T])$, $\eta \in L^2(\Gamma_2 \times (0,T))$, $a_0 \in H$, $\tau_0 \in L^2(\Omega)$, then there exists one and only one weak solution $\{u, \theta\}$ of (1), (2) satisfying the initial condition (3). Furthermore

$$u \in C([0,T] : H), \ \theta \in C([0,T] : L^2(\Omega)).$$

Definition 2

$\{u, \theta\}$ is called a periodic weak solution of (1), (2) with period T_0, if $\{u, \theta\}$ is a weak solution of (1), (2) for $T = T_0$ satisfying

$$u(x, T_0) = u(x, 0), \quad \theta(x, T_0) = \theta(x, 0). \tag{5}$$

We also obtained the existence of periodic weak solutions([3]).

Theorem B

Let Ω be a bounded domain in R^2 with C^1 boundary satisfying Condition (H). Let $g(x,t)$, $\xi(x,t)$, $\eta(x,t)$ be periodic with respect to t with period T_0, satisfying $g \in L^\infty(\Omega \times (0, T_0))$, $\xi \in C^1(\overline{\Gamma}_1 \times [0, T_0])$ and $\eta \in L^2(\Gamma_2 \times (0, T_0))$. Set $g_\infty = ||g||_{L^\infty(\Omega \times (0,T_0))}$. If $\dfrac{\beta g_\infty}{\sqrt{\nu\chi}}$ is sufficiently small, then there exists a periodic weak solution of (1), (2) with period T_0. Furthermore

$$u \in C([0,\infty) : H), \ \theta \in C([0,\infty) : L^2(\Omega)).$$

Now we can state our results. As for the uniqueness of periodic weak solutions, we obtained:

Theorem 1

Let $\{u_\pi, \theta_\pi\}$ be a weak periodic solution of (1), (2) with period T_0 such that for some $p > 2$,

$$\text{ess.sup}_t\{c\|u_\pi(t)\|_p + \frac{1}{4\chi}(c\|\theta_\pi(t)\|_p + c'\beta g_\infty)^2\} < \nu, \tag{6}$$

where c and c' are constants depending on Ω. If $\{u_\pi + u, \theta_\pi + \theta\}$ is a weak periodic solution of (1), (2) with period T_0, then $u = 0, \theta = 0$.

Let $g \in L^\infty(\Omega \times (0,\infty))$, $\xi \in C^1(\overline{\Gamma}_1 \times [0,\infty))$ $\eta \in L^2(\Gamma_2 \times (0,\infty))$, $a_0 \in H, \tau_0 \in L^2(\Omega)$. Let T be any positive number. Then there exists one and only one weak solution $\{u_T, \theta_T\}$ of (1), (2) satisfying (3). Therefore, for $T < T'$,

$$u_T(t) = u_{T'}(t), \quad \theta_T(t) = \theta_{T'}(t) \quad \text{for } \forall t \in (0,T)$$

hold, and we can omit T. This solution is called a global weak solution. We obtained the asymptotic property of solutions of Boussinesq equations as follows.

Theorem 2

Let g, ξ, η satisfy the condition of Theorem B, $a_0 \in H$, $\tau_0 \in L^2(\Omega)$. Let $\{u, \theta\}$ be a global weak solution of (1), (2) satisfying (3), $\{u_\pi, \theta_\pi\}$ a periodic weak solution satisfying (6). Then

$$\lim_{t\to\infty}\{\|u(t) - u_\pi(t)\|^2 + \|\theta(t) - \theta_\pi(t)\|^2\} = 0.$$

Remark

(i) Since $u_\pi \in L^2(0,T : V) \cap C([0,T] : H)$, u_π belongs to the space $L^{2p/(p-2)}(0,T : \mathbf{L}^p(\Omega))$ for $\forall p > 2$. Similarly θ_π is in $L^{2p/(p-2)}(0,T : L^p(\Omega))$. The condition (6) is stronger than this one.

(ii) When (6) holds, such periodic solution is unique (Theorem 1).

References

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